6.5620 Foundations of Cryptography
Lecture 25

Program Obfuscation

and the Quest for Cryptography's Holy Grail

Program Obfuscation

Obfuscation

n. the action of making something obscure, unclear, or unintelligible.

```
#include<stdio.h> #include<string.h>
main(){char*O,l[999]="'`acgo\177~|xp .
-\OR^8)NJ6%K4O+A2M(*OID57$3G1FBL";
while(O=fgets(l+45,954,stdin)){*l=O[
strlen(O)[O-1]=0,strspn(O,l+11)];
while(*O)switch((*l&&isalnum(*O))-!*l)
{case-1:{char*I=(O+=strspn(O,l+12)
+1)-2,O=34;while(*I&3&&(O=(O-16<<1)+
*I---'-')<80);putchar(O&93?*I
&8||!(  I=memchr( l , O , 44 ) ) ?'?':
I-l+47:32); break; case 1: ;}*l=
(*O&31)[l-15+(*O>61)*32];while(putchar
(45+*l%2),(*l=*l+32>>1)>35); case 0:
putchar((++O ,32));}putchar(10);}}
```

Figure 1: The winning entry of the 1998 *International Obfuscated C Code Contest*, an ASCII/Morse code translator by Frans van Dorsselaer [vD] (adapted for this paper).

```c
#include                                        <math.h>
#include                                      <sys/time.h>
#include                                      <X11/Xlib.h>
#include                                     <X11/keysym.h>
                                             double L ,o ,P
                                            ,_=dt,T,Z,D=1,d,
                                            s[999],E,h= 8,I,
                                            J,K,w[999],M,m,O
                                           ,n[999],j=33e-3,i=
                                           1E3,r,t, u,v ,W,S=
                                           74.5,l=221,X=7.26,
                                           a,B,A=32.2,c, F,H;
                                           int N,q, C, y,p,U;
                                          Window z; char f[52]
                                       ; GC k; main(){ Display*e=
 XOpenDisplay( 0); z=RootWindow(e,0); for (XSetForeground(e,k=XCreateGC (e,z,0,0),BlackPixel(e,0))
; scanf("%lf%lf%lf",y +n,w+y, y+s)+1; y ++); XSelectInput(e,z= XCreateSimpleWindow(e,z,0,0,400,400,
0,0,WhitePixel(e,0) ),KeyPressMask); for(XMapWindow(e,z); ; T=sin(O)){ struct timeval G={ 0,dt*1e6}
; K= cos(j); N=1e4; M+= H*_; Z=D*K; F+=_*P; r=E*K; W=cos( O); m=K*W; H=K*T; O+=D*_*F/ K+d/K*E*_; B=
sin(j); a=B*T*D-E*W; XClearWindow(e,z); t=T*E+ D*B*W; j+=d*_*D-_*F*E; P=W*E*B-T*D; for (o+=(I=D*W+E
*T*B,E*d/K *B+v+B/K*F*D)*_; p<y; ){ T=p[s]+i; E=c-p[w]; D=n[p]-L; K=D*m-B*T-H*E; if(p [n]+w[ p]+p[s
]== 0|K <fabs(W=T*r-I*E +D*P) |fabs(D=t *D+Z *T-a *E)> K)N=1e4; else{ q=W/K *4E2+2e2; C= 2E2+4e2/ K
 *D; N-1E4&& XDrawLine(e ,z,k,N ,U,q,C); N=q; U=C; } ++p; } L+=_* (X*t +P*M+m*l); T=X*X+ l*l+M *M;
  XDrawString(e,z,k ,20,380,f,17); D=v/l*15; i+=(B *l-M*r -X*Z)*_; for(; XPending(e); u *=CS!=N){
                                      XEvent z; XNextEvent(e ,&z);
                                          ++*((N=XLookupKeysym
                                            (&z.xkey,0))-IT?
                                            N-LT? UP-N?& E:&
                                            J:& u: &h); --*(
                                            DN -N? N-DT ?N==
                                            RT?&u: & W:&h:&J
                                             ); } m=15*F/l;
                                             c+=(I=M/ l,l*H
                                             +I*M+a*X)*_; H
                                             =A*r+v*X-F*l+(
                                             E=.1+X*4.9/l,t
                                             =T*m/32-I*T/24
                                              )/S; K=F*M+(
                                              h* 1e4/l-(T+
                                              E*5*T*E)/3e2
                                              )/S-X*d-B*A;
                                              a=2.63 /l*d;
                                              X+=( d*l-T/S
                                               *(.19*E +a
                                               *.64+J/1e3
                                               )-M* v +A*
                                               Z)*_; l +=
                                               K *_; W=d;
                                               sprintf(f,
                                               "%5d  %3d"
                                               "%7d",p =l
                                              /1.7,(C=9E3+
                                 O*57.3)%0550,(int)i); d+=T*(.45-14/l*
                                X-a*130-J* .14)*_/125e2+F*_*v; P=(T*(47
                                *I-m* 52+E*94 *D-t*.38+u*.21*E) /1e2+W*
                                179*v)/2312; select(p=0,0,0,0,&G); v-=(
                                 W*F-T*(.63*m-I*.086+m*E*19-D*25-.11*u
                                  )/107e2)*_; D=cos(o); E=sin(o); } }
```

```c
#define _ F-->00 || F-OO--;
long F=00,OO=00;
main(){F_OO();printf("%1.3f\n", 4.*-F/OO/OO);}F_OO()
{
            _-_-_-_
       _-_-_-_-_-_-_-_-_
    _-_-_-_-_-_-_-_-_-_-_-_
  _-_-_-_-_-_-_-_-_-_-_-_-_-_
 _-_-_-_-_-_-_-_-_-_-_-_-_-_-_
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  _-_-_-_-_-_-_-_-_-_-_-_-_-_
    _-_-_-_-_-_-_-_-_-_-_-_
       _-_-_-_-_-_-_-_-_
            _-_-_-_
}
```

Answer: Run me!

Program Obfuscation

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n. the action of making a program unintelligible, while preserving its input/output behavior.

Program Obfuscation

Programs that contain secrets: {

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- Watermarks
- Trapdoors
- The Algorithm itself

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Hmm… I am going out of town but would like to delegate all 6.875 emails to my TAs.

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def DecryptEmail(EncryptedEmail):

- sk = "786fe0974effa30621"
- m = Decrypt(EncryptedEmail, sk)
- if m.find("6.875"), return m
- Else, return "Sorry, this e-mail is private"

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```

138805012AA98B7920FC10385089012408A292E0
0FF0016590090165 9AA1606B692650F3893EE390
30957BE927A6789C10846DD10AA92DEADBEEF
09179578134

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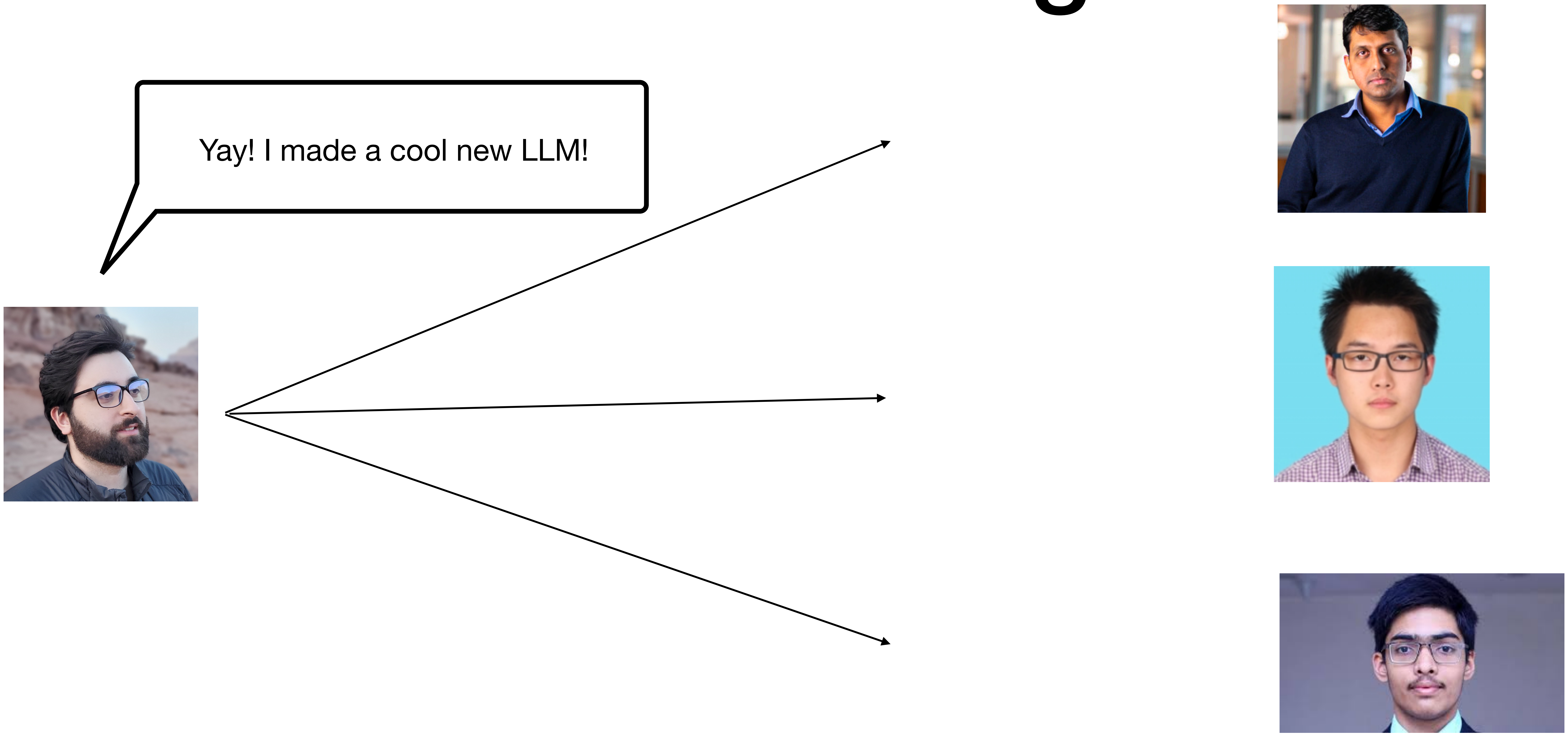
- Else, return “Sorry, this e-mail is private”

Watermarking

Yay! I made a cool new LLM!

Watermarking



Watermarking

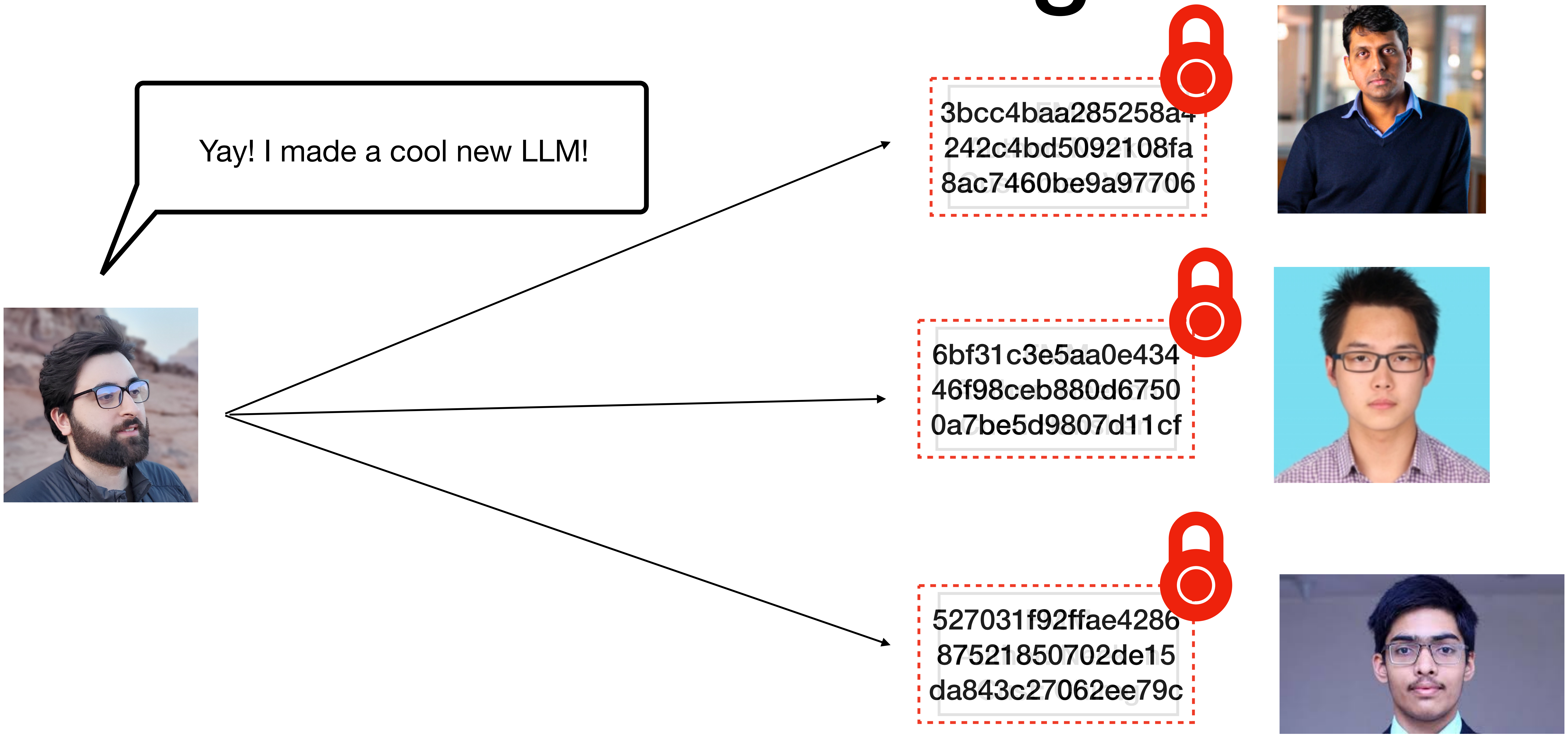
Yay! I made a cool new LLM!

FMM
Author: Neekon
Customer: Vinod

FMM
Author: Neekon
Cust: Hanshen

FMM
Author: Neekon
Cust: Chirag

Watermarking



Watermarking

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3bcc4baa285258a4
242c4bd5092108fa
8ac7460be9a97706

6bf31c3e5aa0e434
46f98ceb880d6750
0a7be5d9807d11cf

527031f92ffae4286
87521850702de15
da843c27062ee79c

The watermarks are
now difficult to remove!

The algorithm itself!

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$1 million

Program
Runtime: $O(n)$

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Algorithm with $O(n^3)$ dummy steps

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Program
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$10k

Program
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$1

Program
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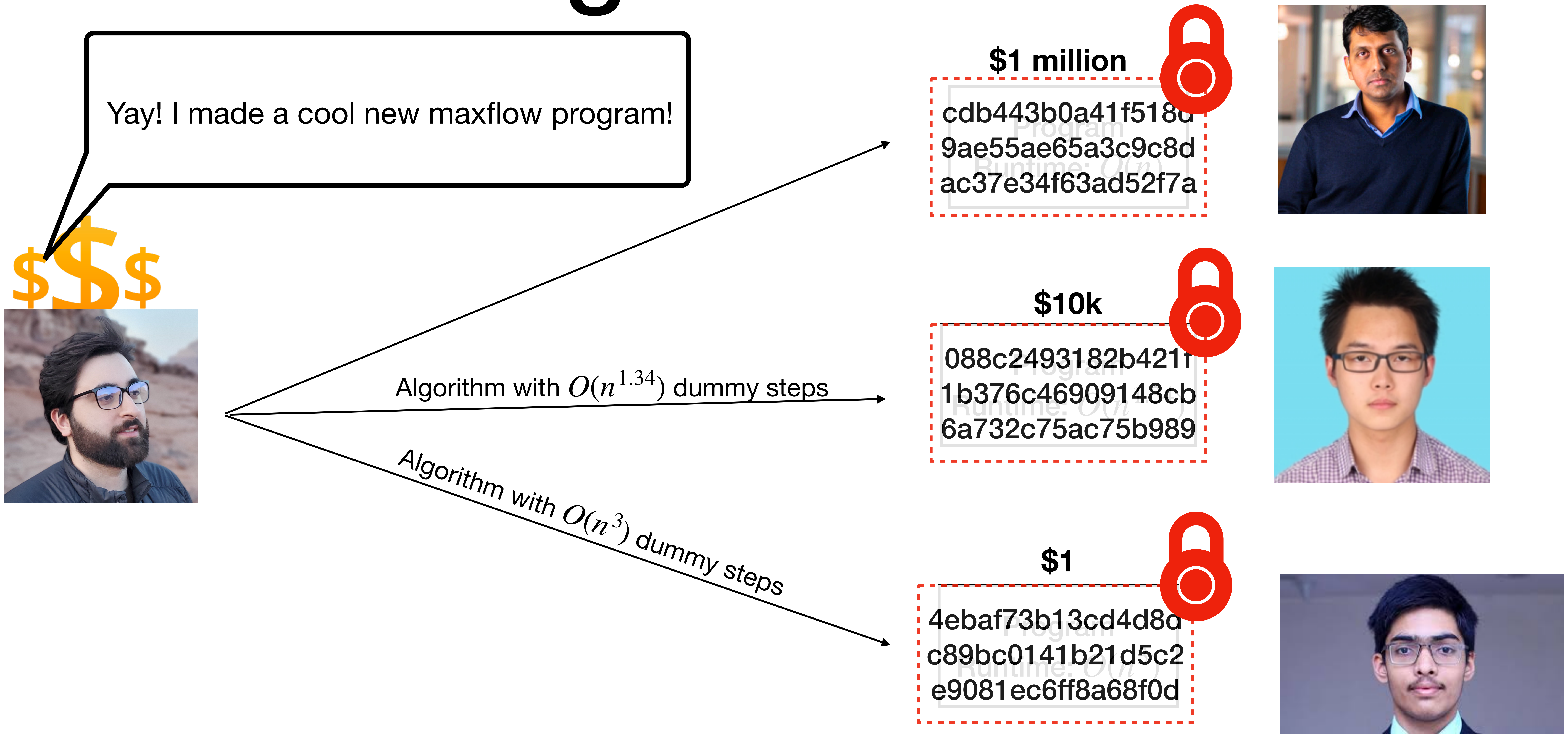
The algorithm itself!



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cdb443b0a41f518d
9ae55ae65a3c9c8d
ac37e34f63ad52f7a

$10k

088c2493182b421f
1b376c46909148cb
6a732c75ac75b989

$1

4ebaf73b13cd4d8d
c89bc0141b21d5c2
e9081ec6ff8a68f0d

This makes the fast algorithm hard to extract!

How to define program obfuscation

Virtual Black-Box

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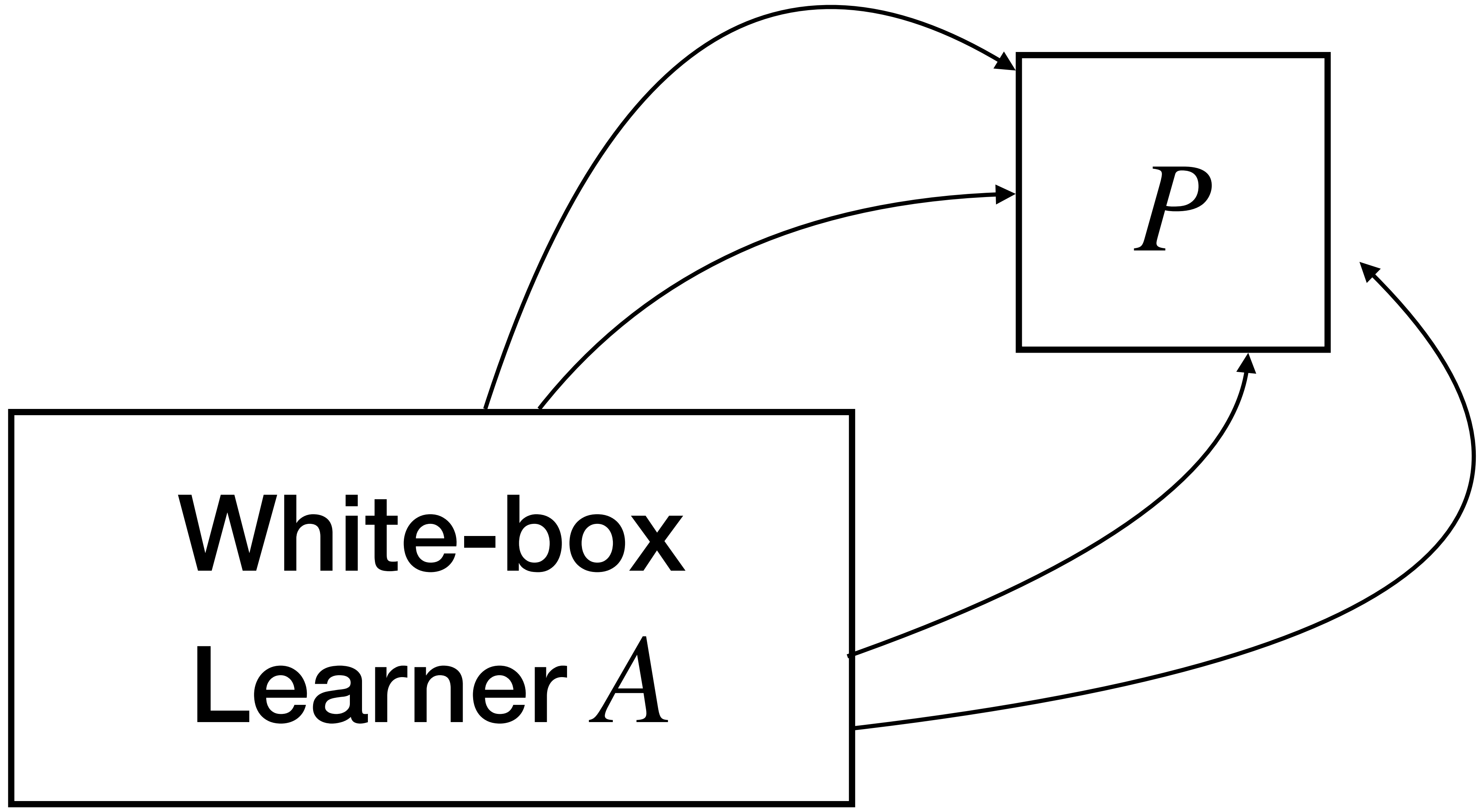

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For all white-box learners A, there exists a simulator S such that for all predicates P

$$|\Pr[A(\mathcal{O}(C)) = P(C)] - \Pr[S^C(1^{|C|}) = P(C)]| \leq \mathsf{negl}(|C|)$$

PKE from SKE!

Secret-Key Encryption

Enc_{sk}

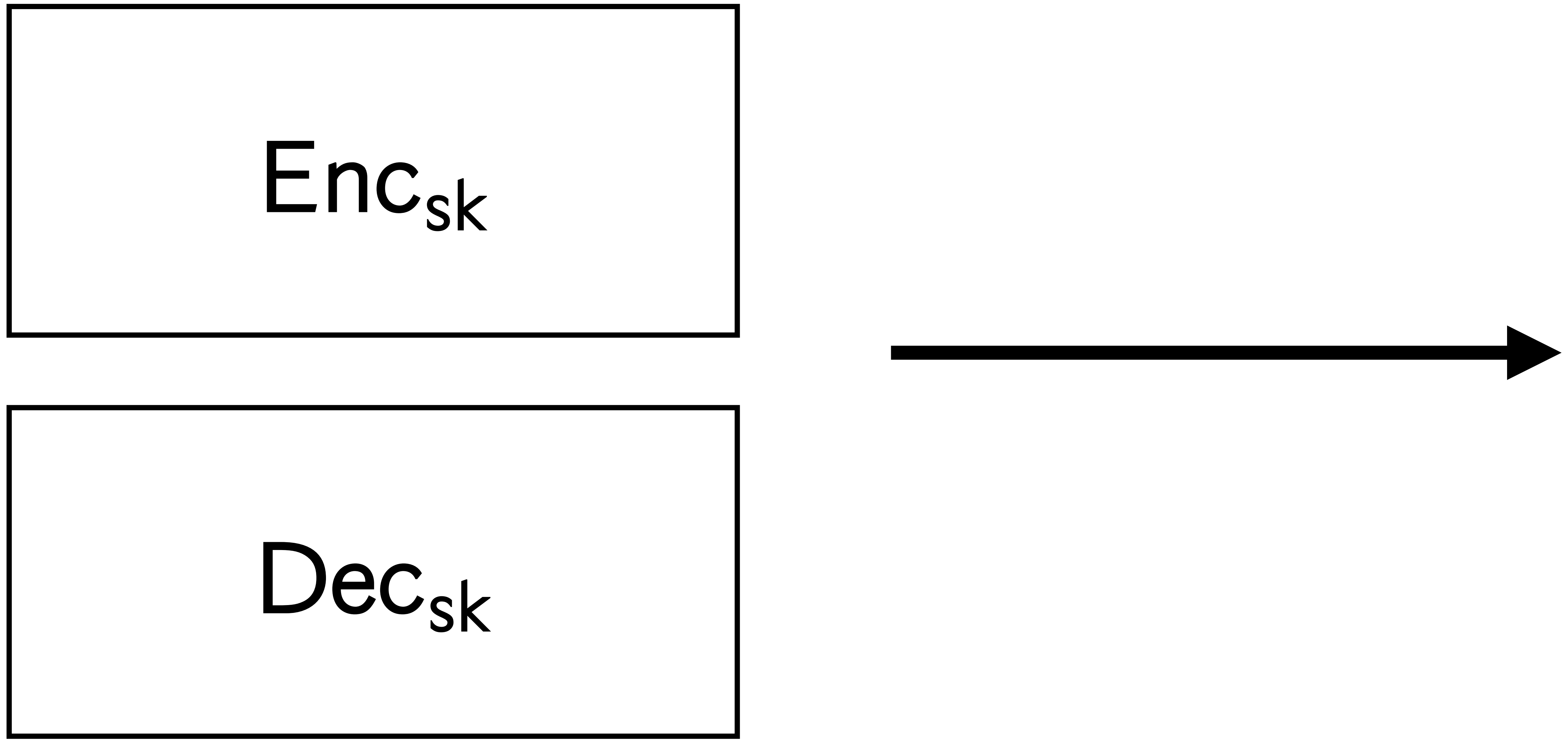
Dec_{sk}

Public-Key Encryption!

PKE from SKE!

Secret-Key Encryption

Public-Key Encryption!



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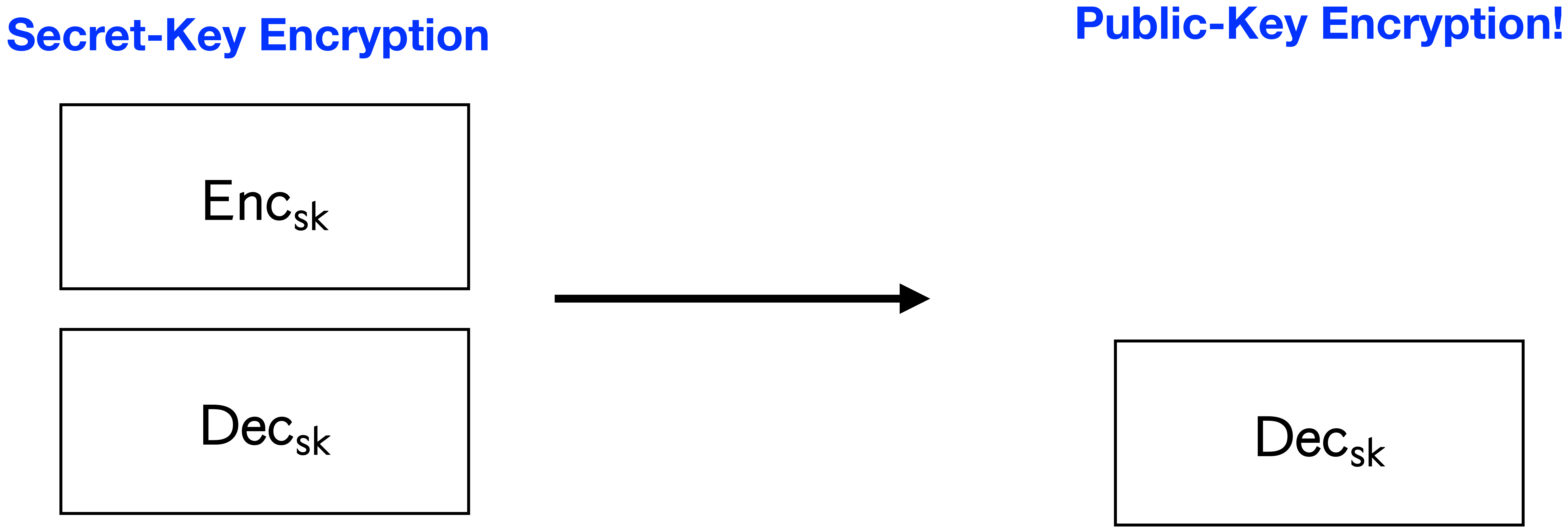
Secret-Key Encryption
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Enc$_{sk}$
Dec$_{sk}$
Dec$_{sk}$

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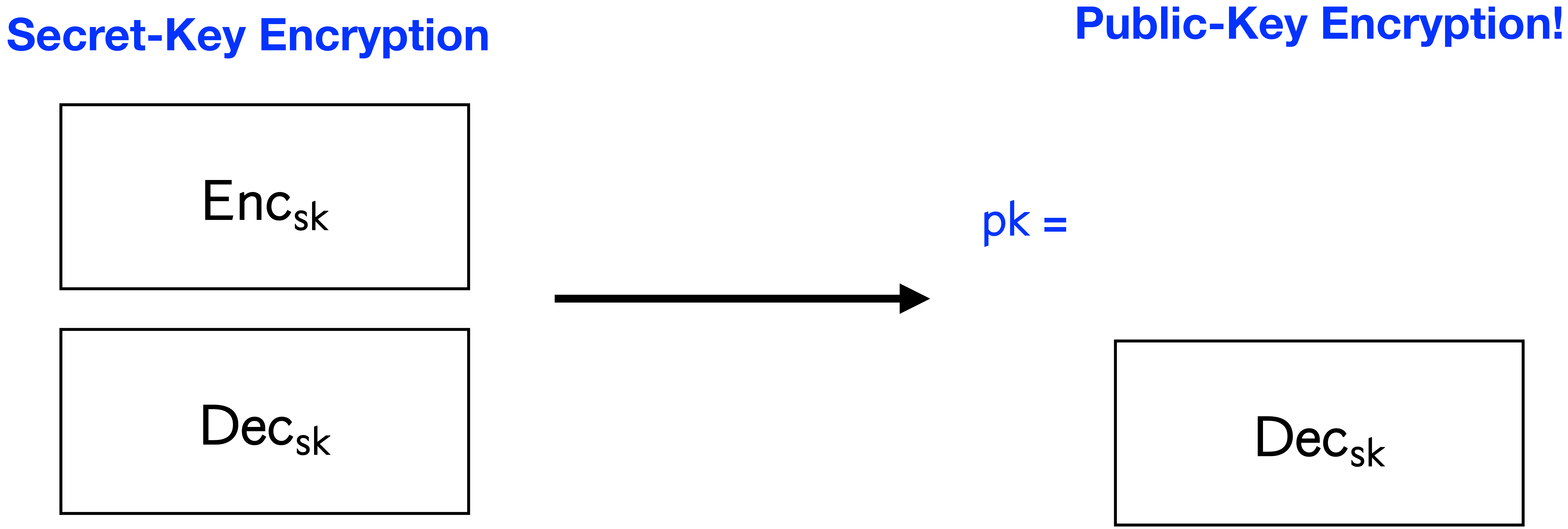

Secret-Key Encryption
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Dec_{sk}
pk =
Dec_{sk}

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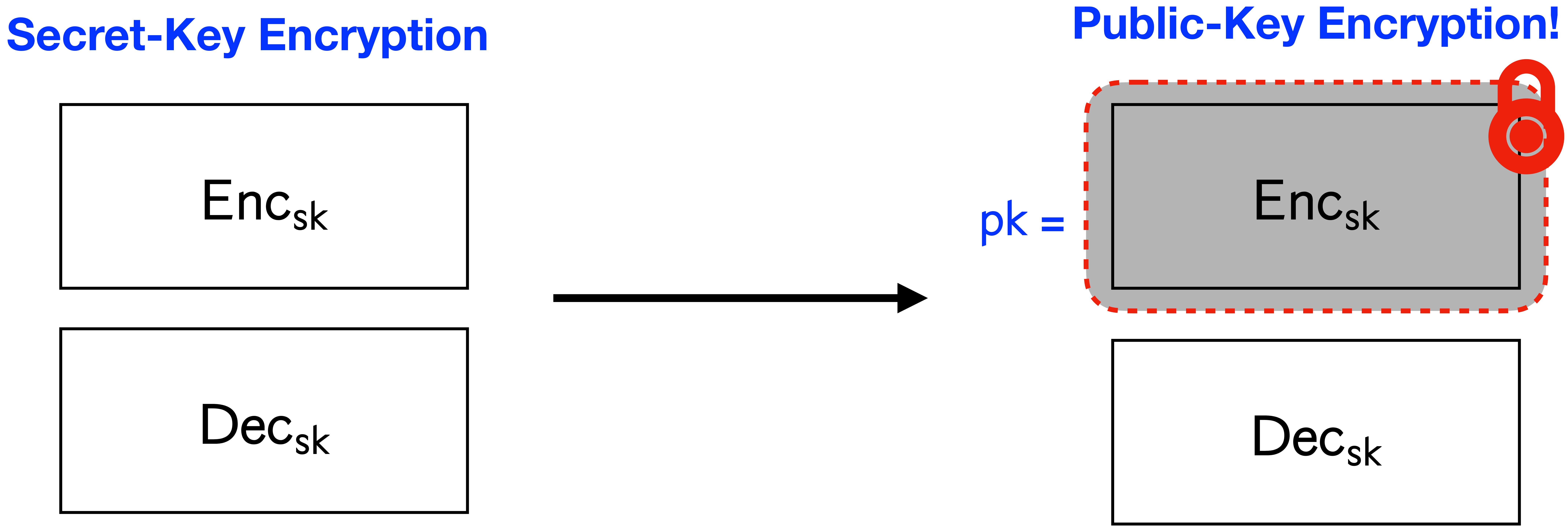
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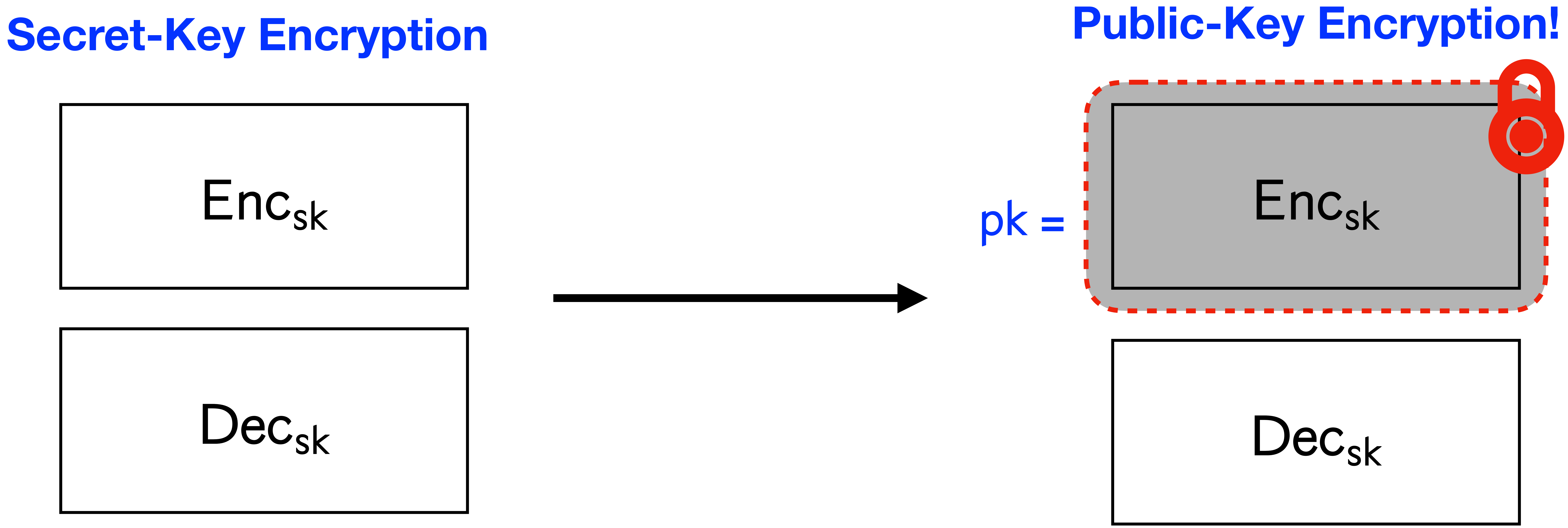

OWF + VBB gives public key encryption!!

Note that we previously used LWE, DDH, RSA, QR etc. to obtain PKE directly.

Diffie-Hellman (1976)

Essentially what is required is a one-way compiler: one which takes an easily understood program written in a high level language and translates it into an incomprehensible program in some machine language. The compiler is one-way because it must be feasible to do the compilation, but infeasible to reverse the process. Since efficiency in size of program and run time are not crucial in this application, such compilers may be possible if the structure of the machine language can be optimized to assist in the confusion.

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OWF + VBB also gives you FHE!!

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HomEval(c_1, c_2, op)

- $m_1 = \mathsf{Dec}_{\mathsf{sk}}(c_1)$ and $m_2 = \mathsf{Dec}_{\mathsf{sk}}(c_2)$
- $m_3 = m_1 \ \mathsf{op} \ m_2$
- Return $\mathsf{Enc}_{\mathsf{sk}}(m_3)$.

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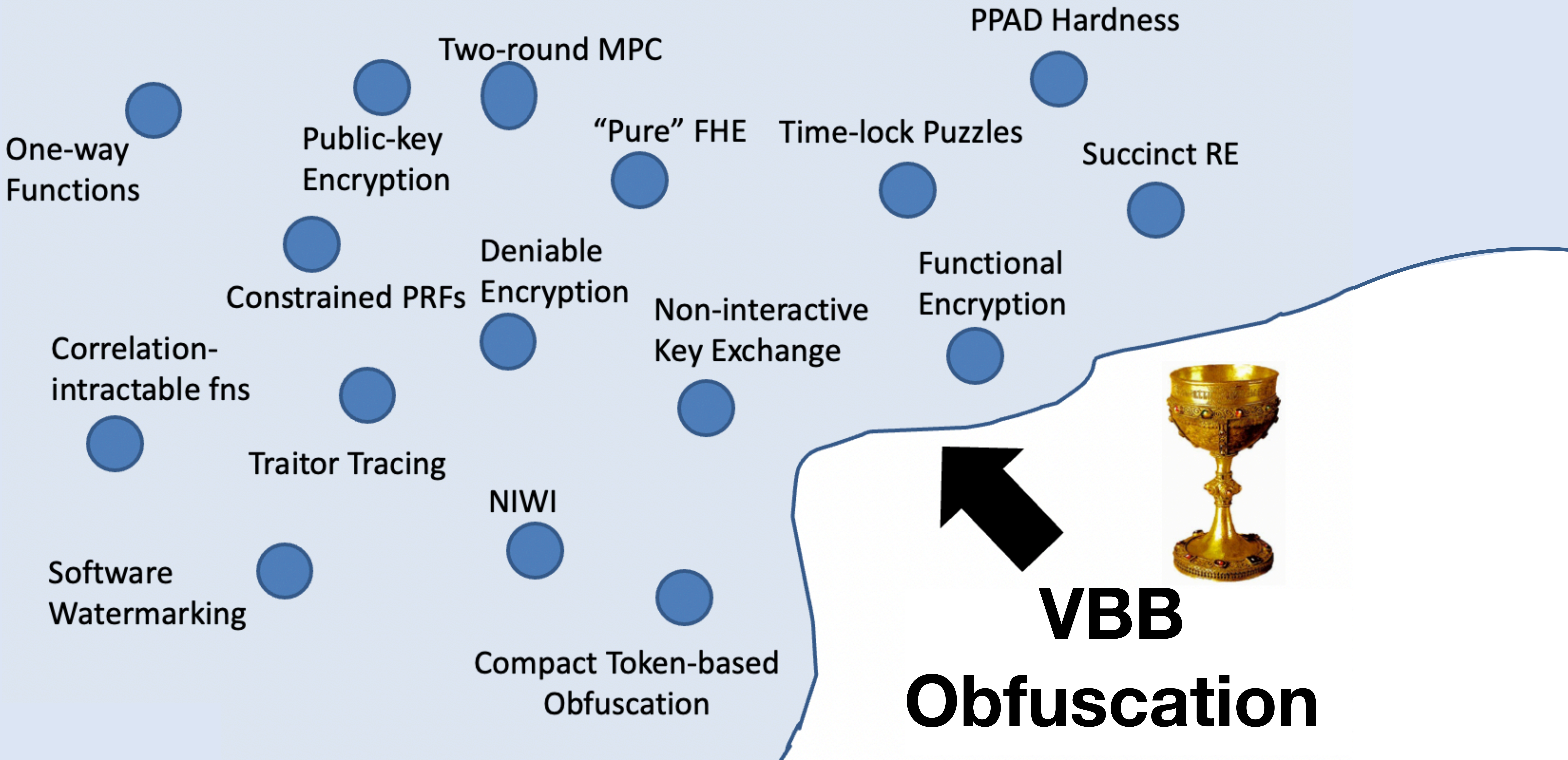
"CRYPTO-COMPLETE":

Nearly all crypto is an easy corollary of VBB!



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Define a family of programs $\mathscr{P} = \{P_{x,y}\}$ where x and y are n-bit strings as follows:

$$P_{x,y}(b,\Pi) = \begin{cases} y & \text{if } b = 0,\ \Pi = x \\ x, y & \text{if } b = 1 \text{ and } \Pi(0,x) = y \\ 0 & \text{otherwise.} \end{cases}$$

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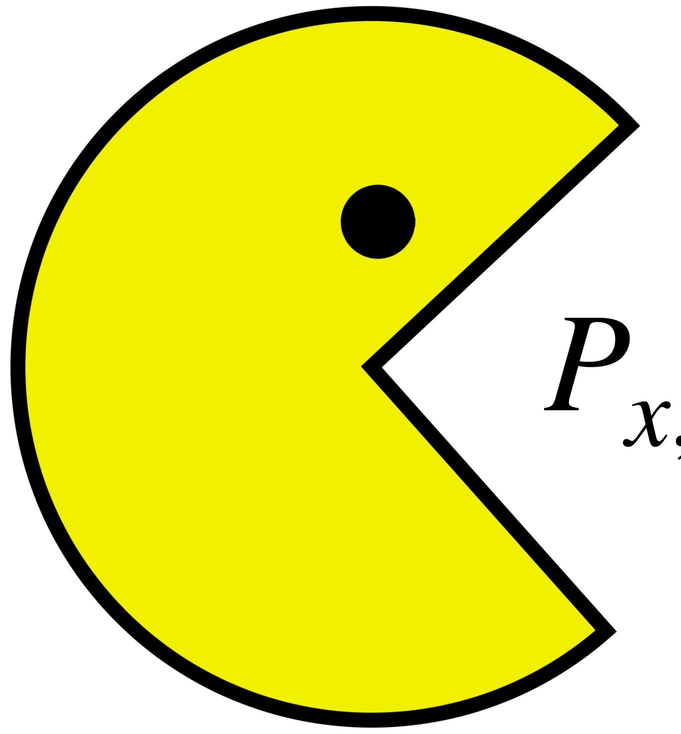

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Remarks

- **One interpretation:** Having code is more powerful than having black-box access: you can run the code with itself as input!
- Proof tells us that even inefficient obfuscates do not exist!
- Can be extended to construct unobfuscatable encryption/signature schemes.

What now?

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Weaken the definition!

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Compare iO vs VBB

Virtual Black-Box Obfuscation

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- **(Indistinguishability)**
  For all pairs C_0 and C_1 computing the same function:
  $$i\mathcal{O}(C_0;r) \approx_c i\mathcal{O}(C_1;r)$$

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Proof:

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Proof:

For any circuit C, let $i\mathcal{O}$ of C be the **lexicographically first circuit** C' that computes the same function as C. Then, we have found a **canonical** representation of each C.

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This can be done in P if P = NP.

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Remark 1: In fact, one can think of iO as a "pseudo-canonicaliser".

Remark 2: This fact means that it is hard to show iO implies OWF (if it did, P $\neq$ NP).

Best possible obfuscation

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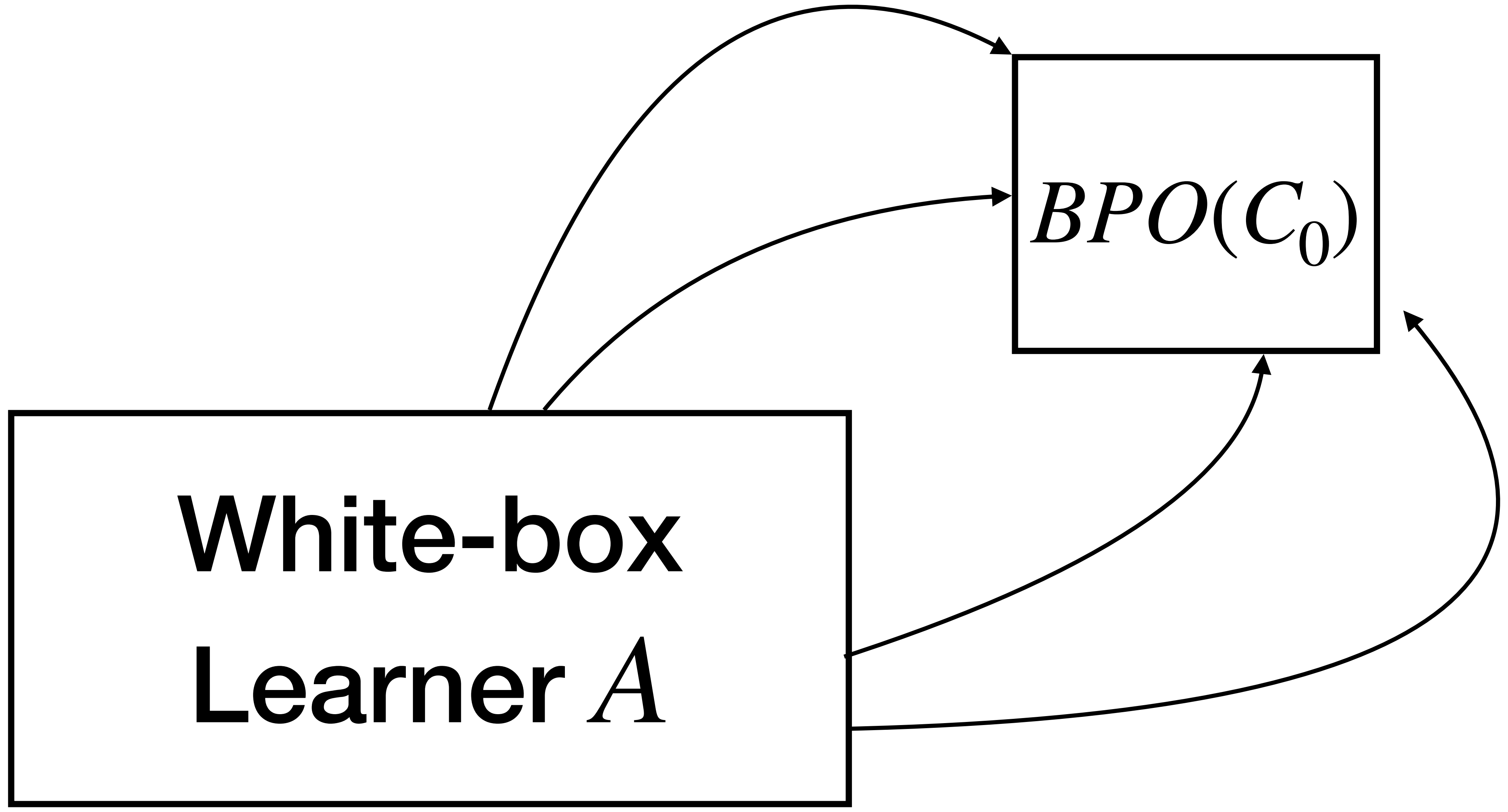

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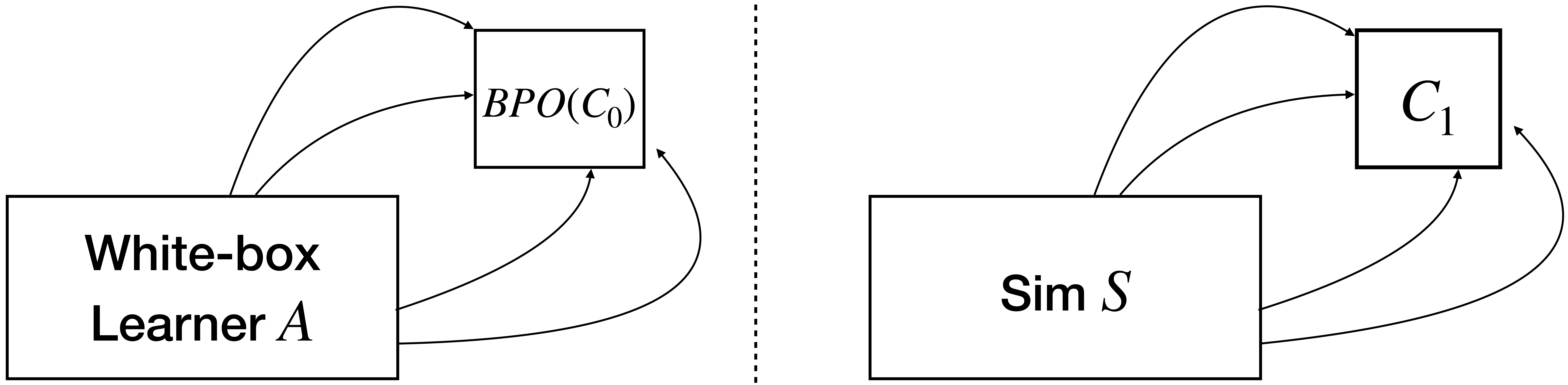

For all white-box learners A, there exists a simulator S such that for all circuits $C_0 \equiv C_1$:

$$|\Pr[A(BPO(C_0)) = 1] - \Pr[S(C_1) = 1]| \leq \text{negl}(|C|)$$

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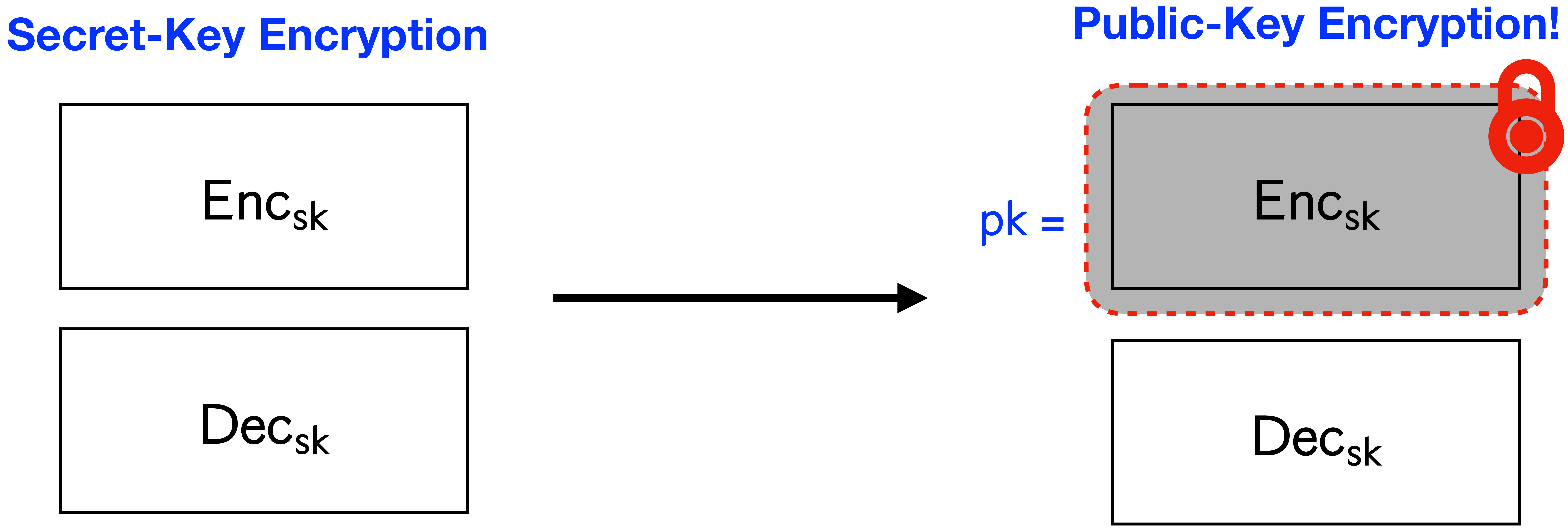
Corollary. If a circuit family has VBB obfuscation, then iO is a VBB obfuscation for this family.

Ok… But what can you do with iO?

iO Gymnastics

Recall: PKE from SKE!



OWF + VBB gives public key encryption!!

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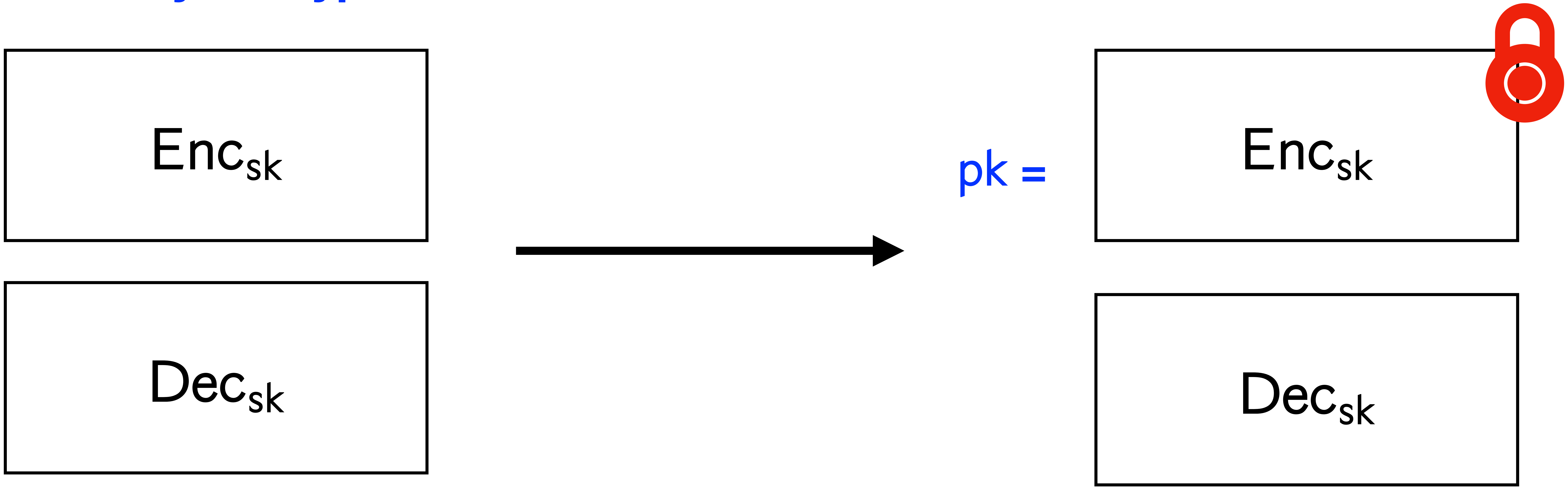
Secret-Key Encryption



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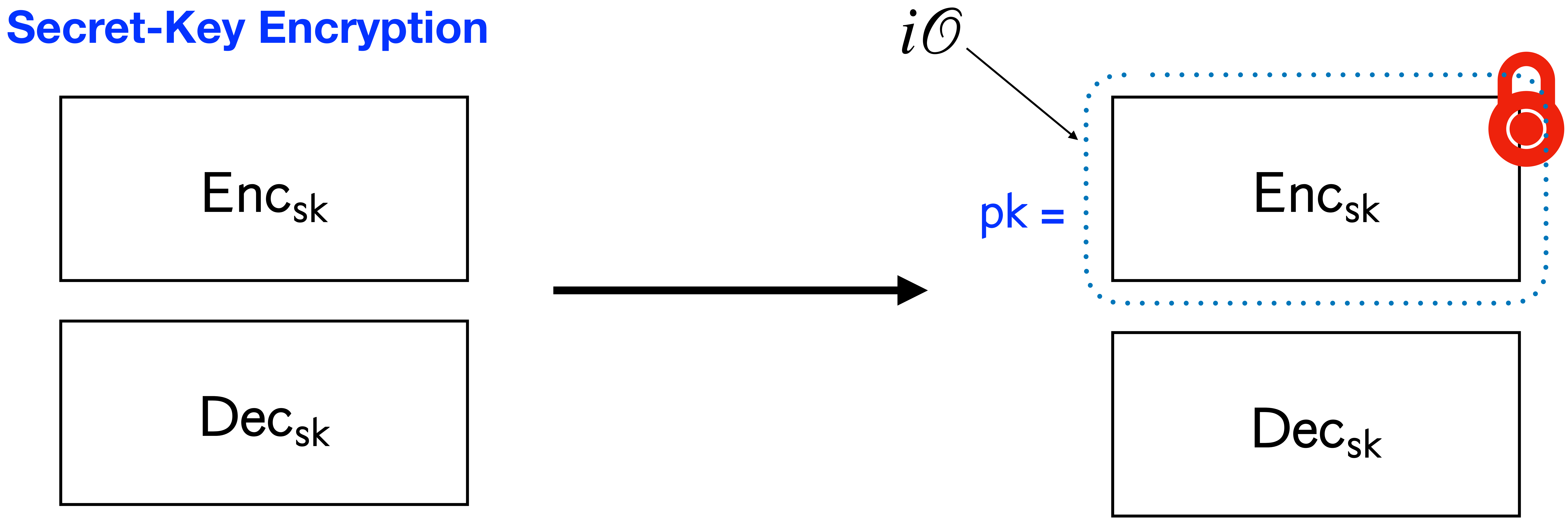

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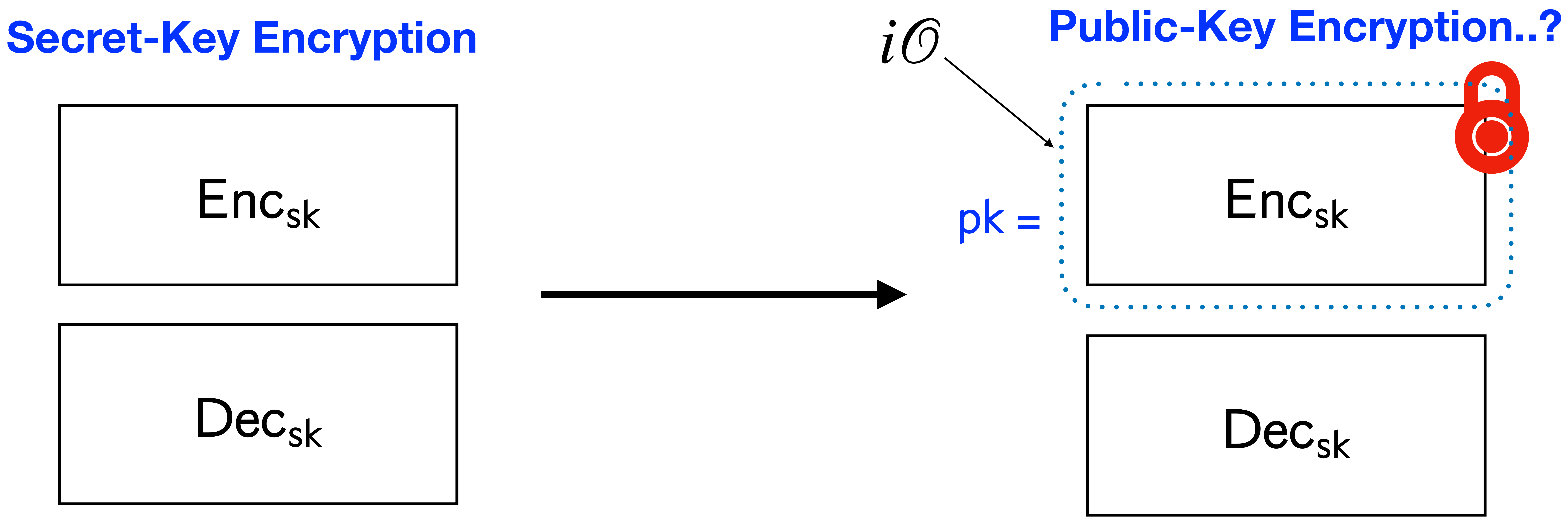

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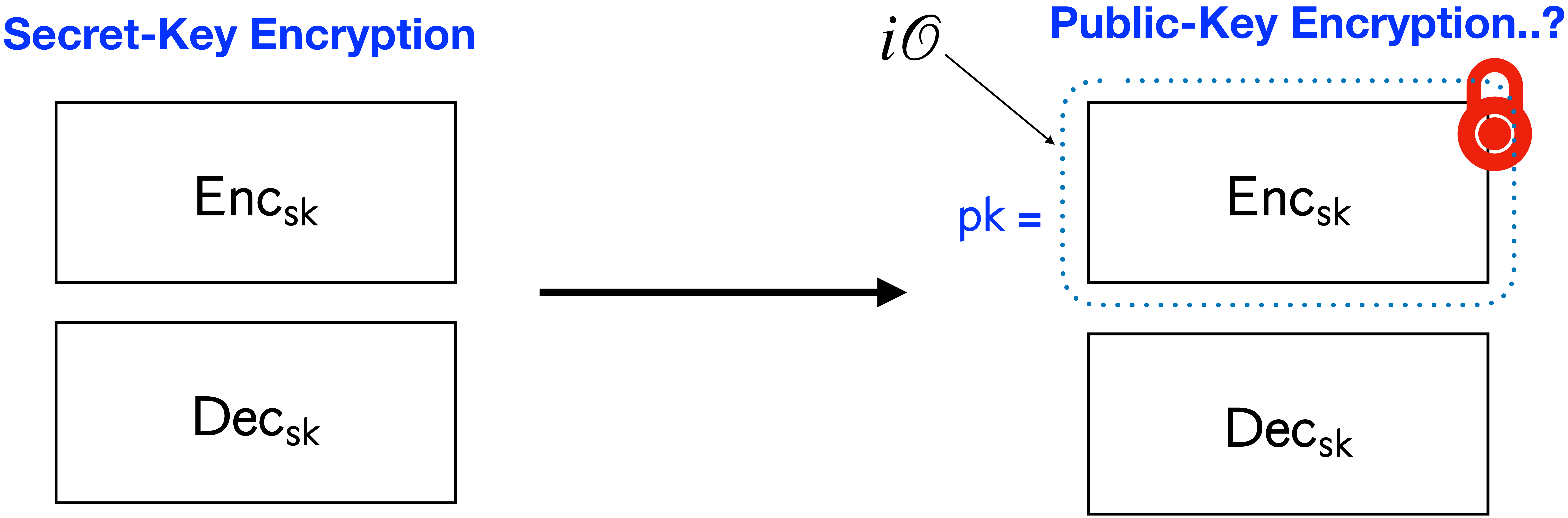

But… iO doesn't really gives us much to work with in this picture…

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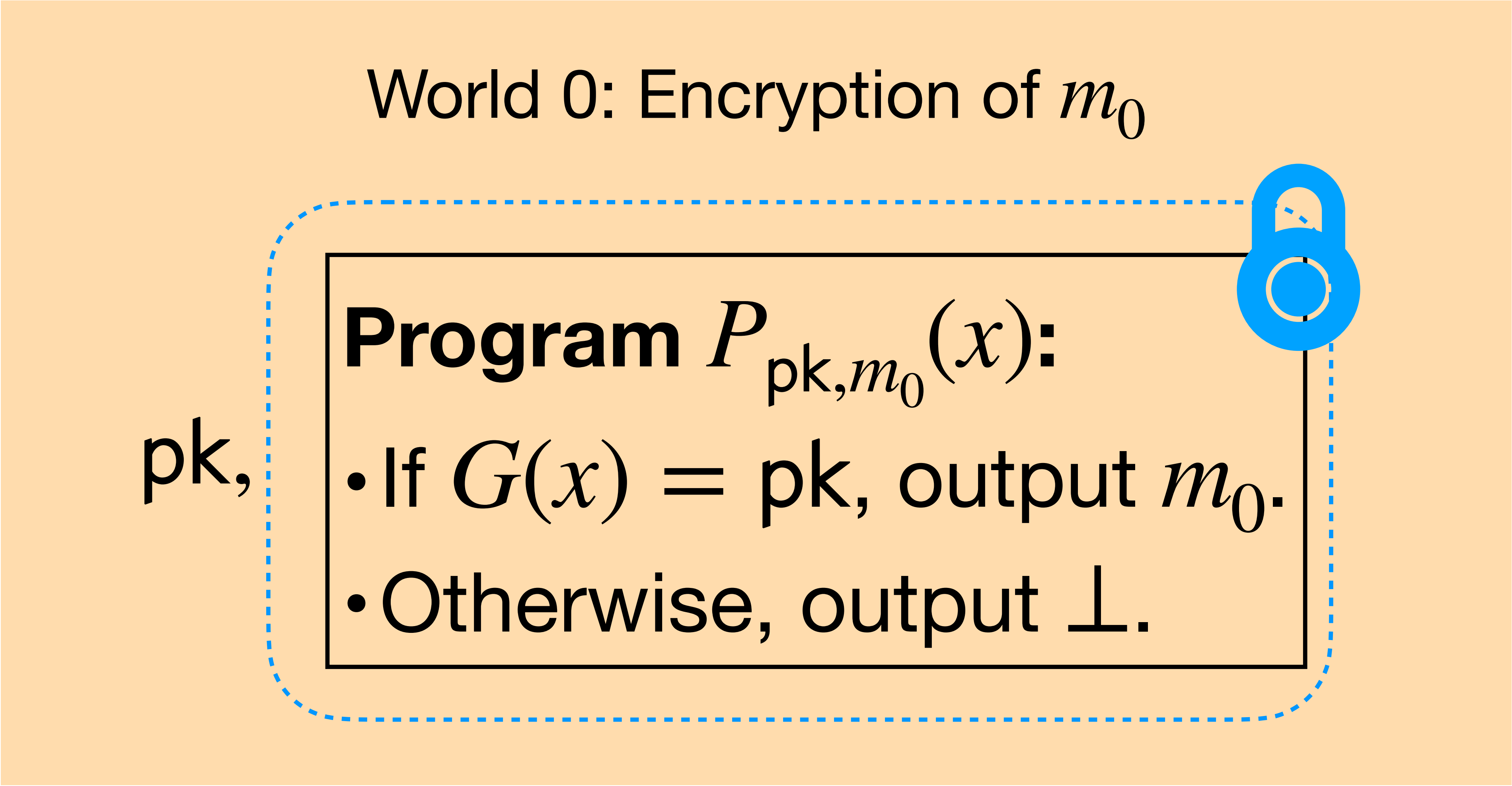

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$\approx_c^?$

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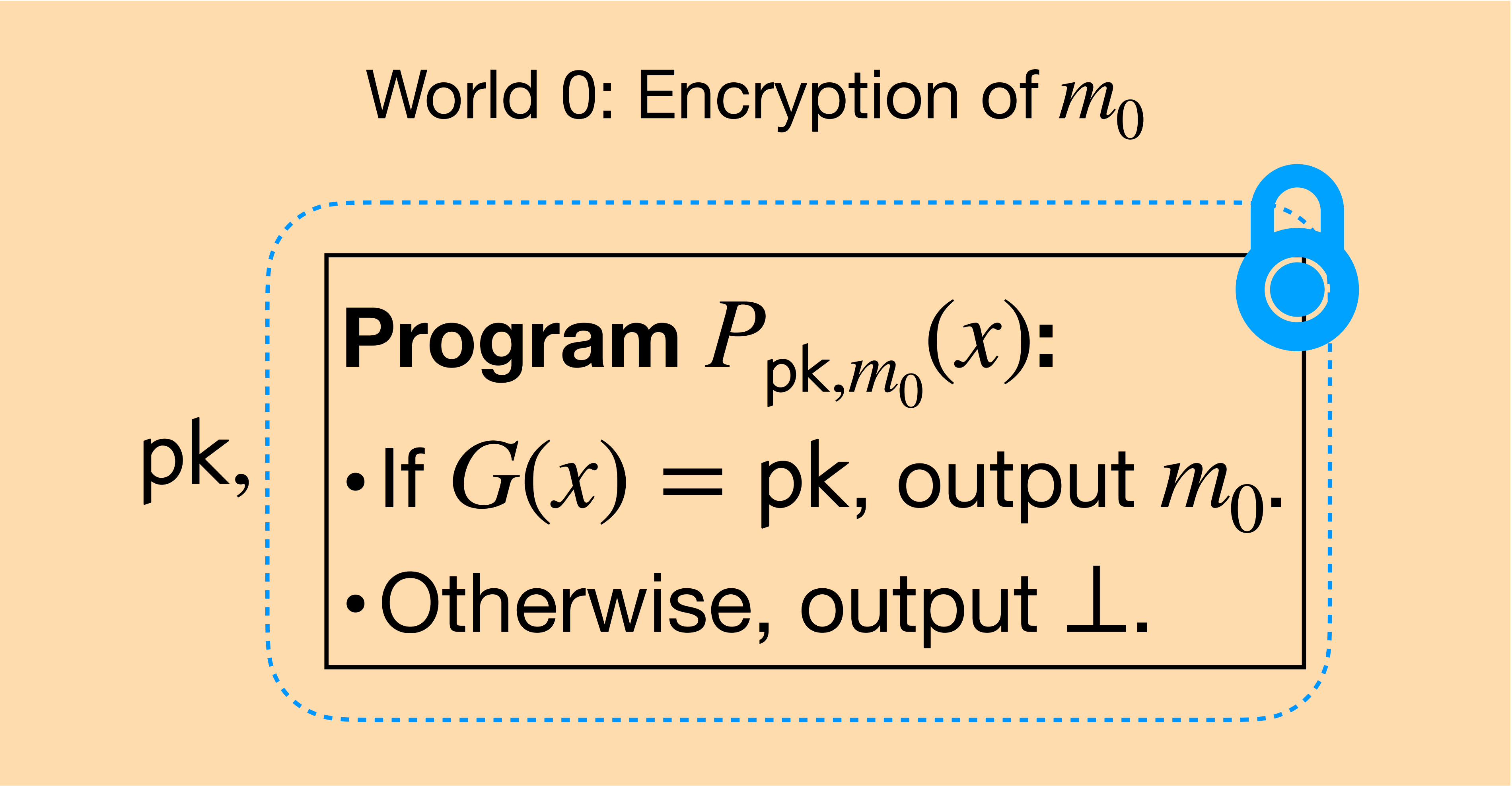
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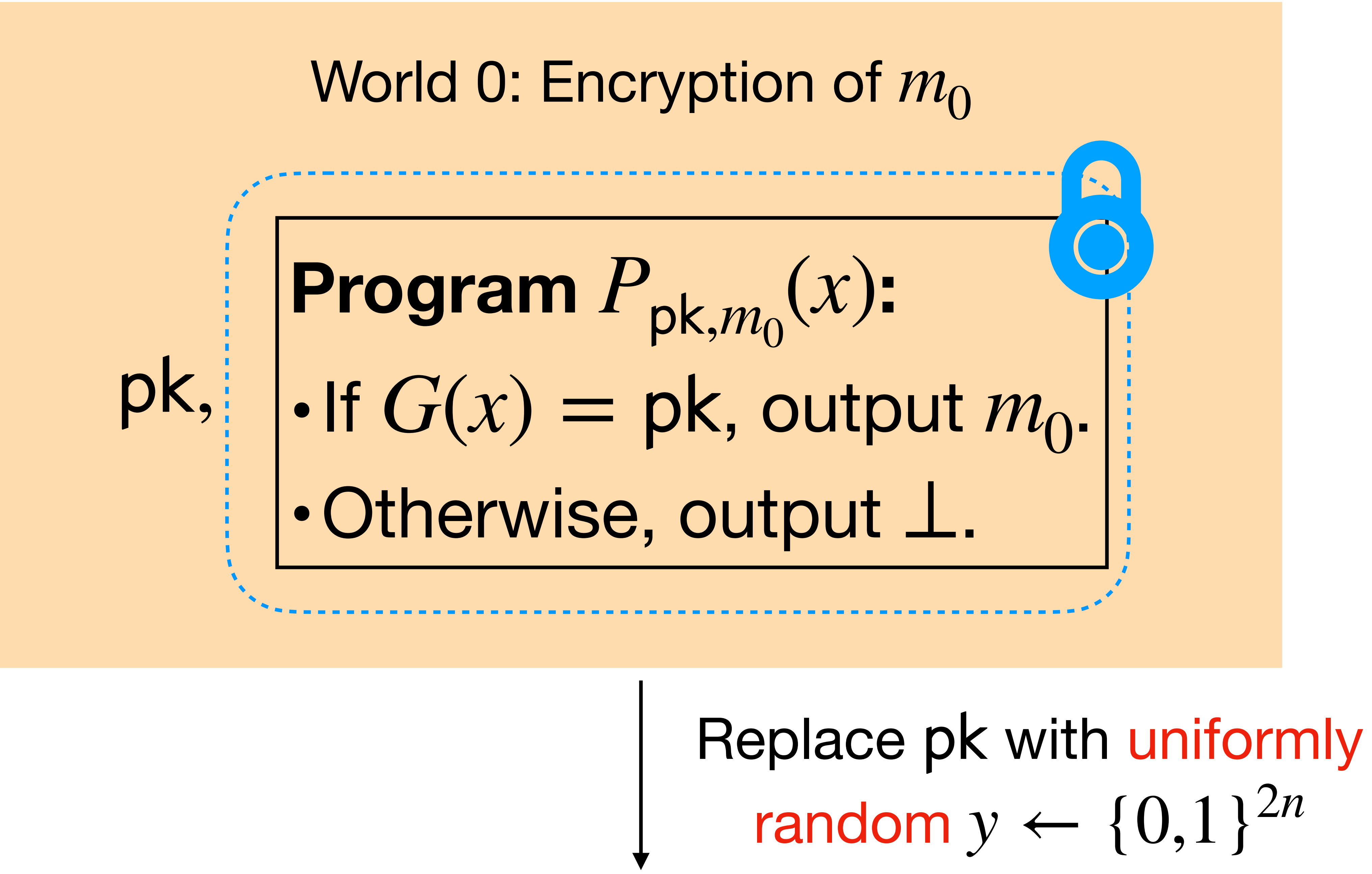
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If an adversary can distinguish these hybrids, then he breaks PRG security!

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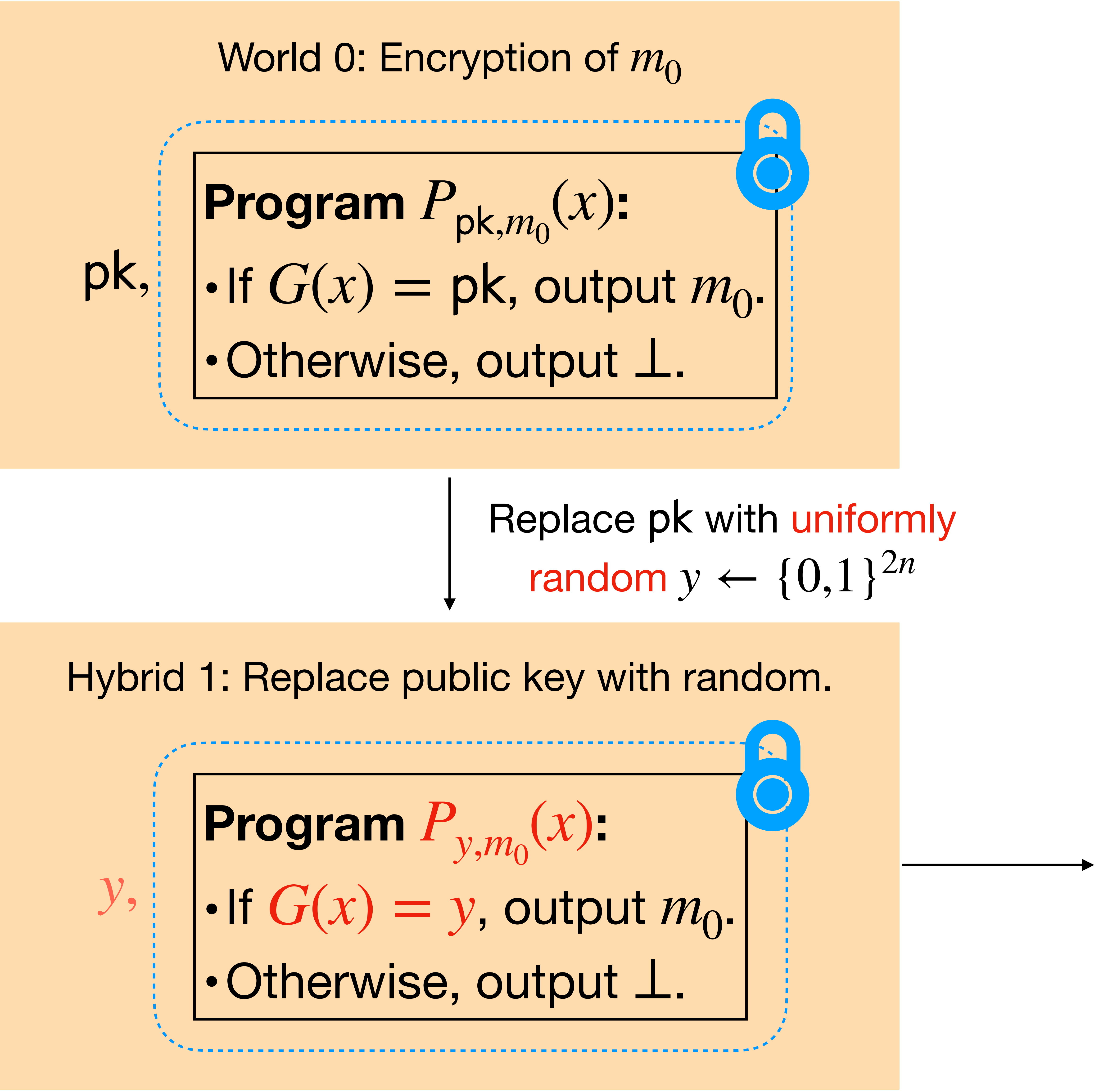

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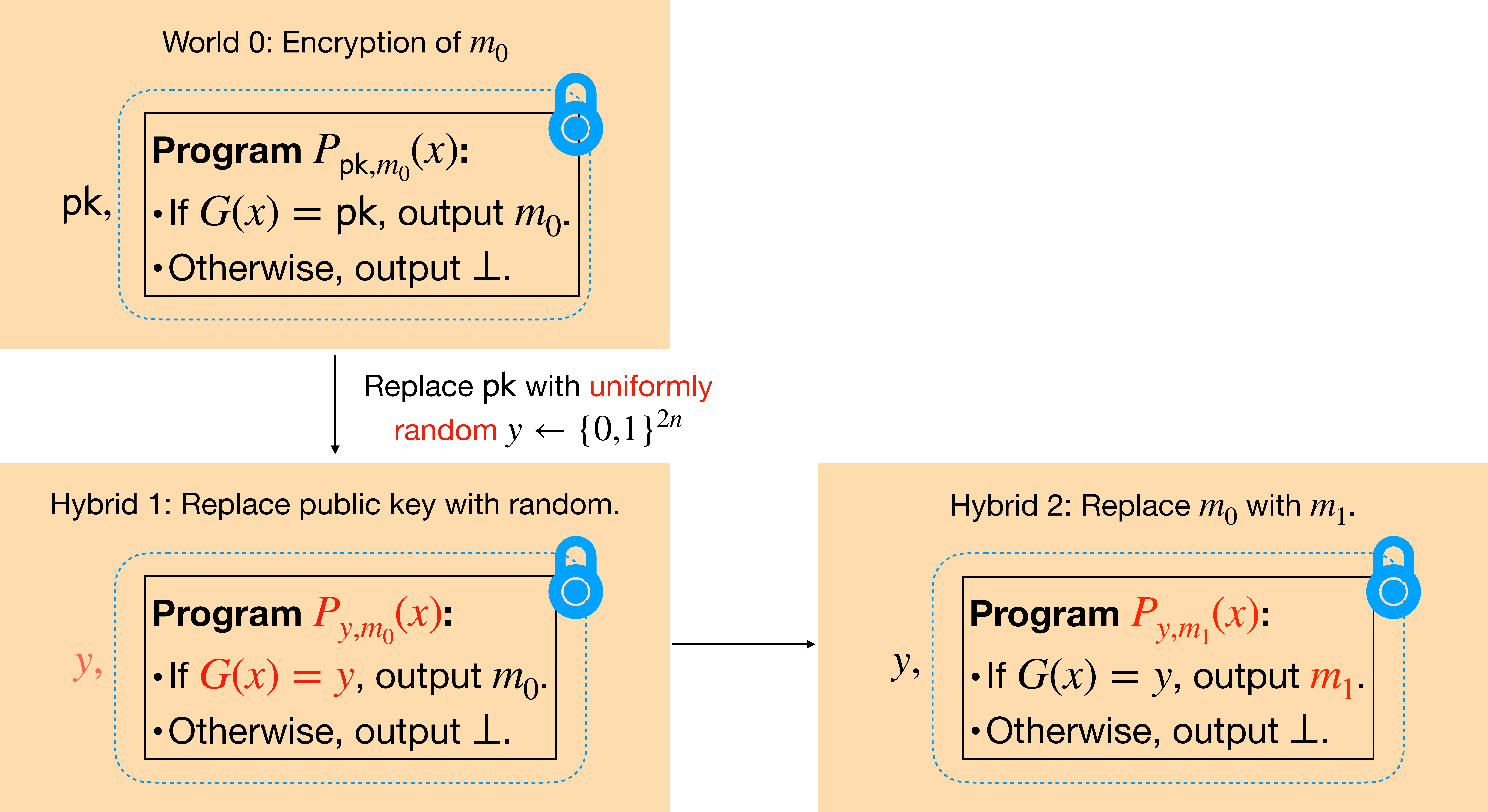
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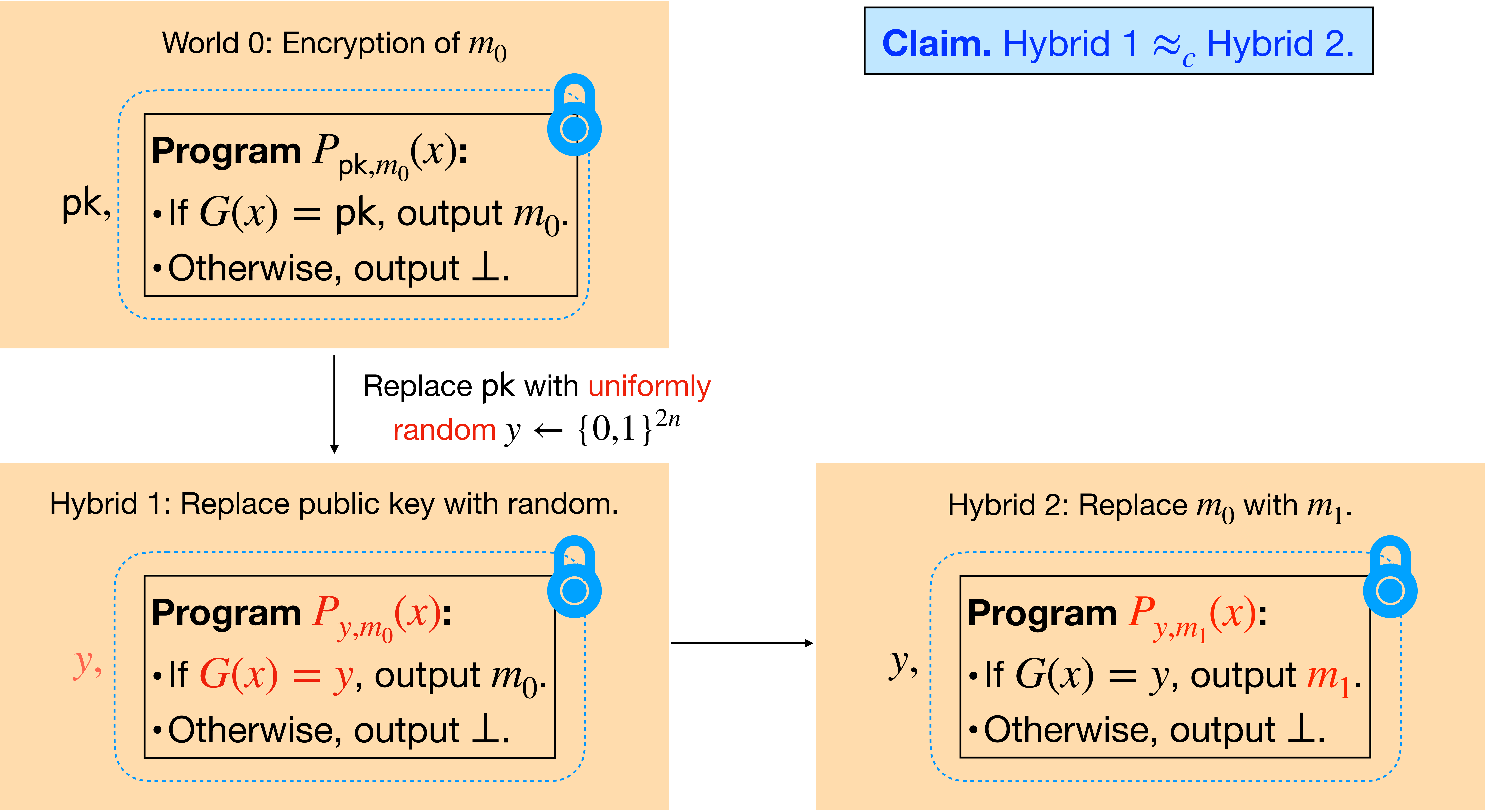

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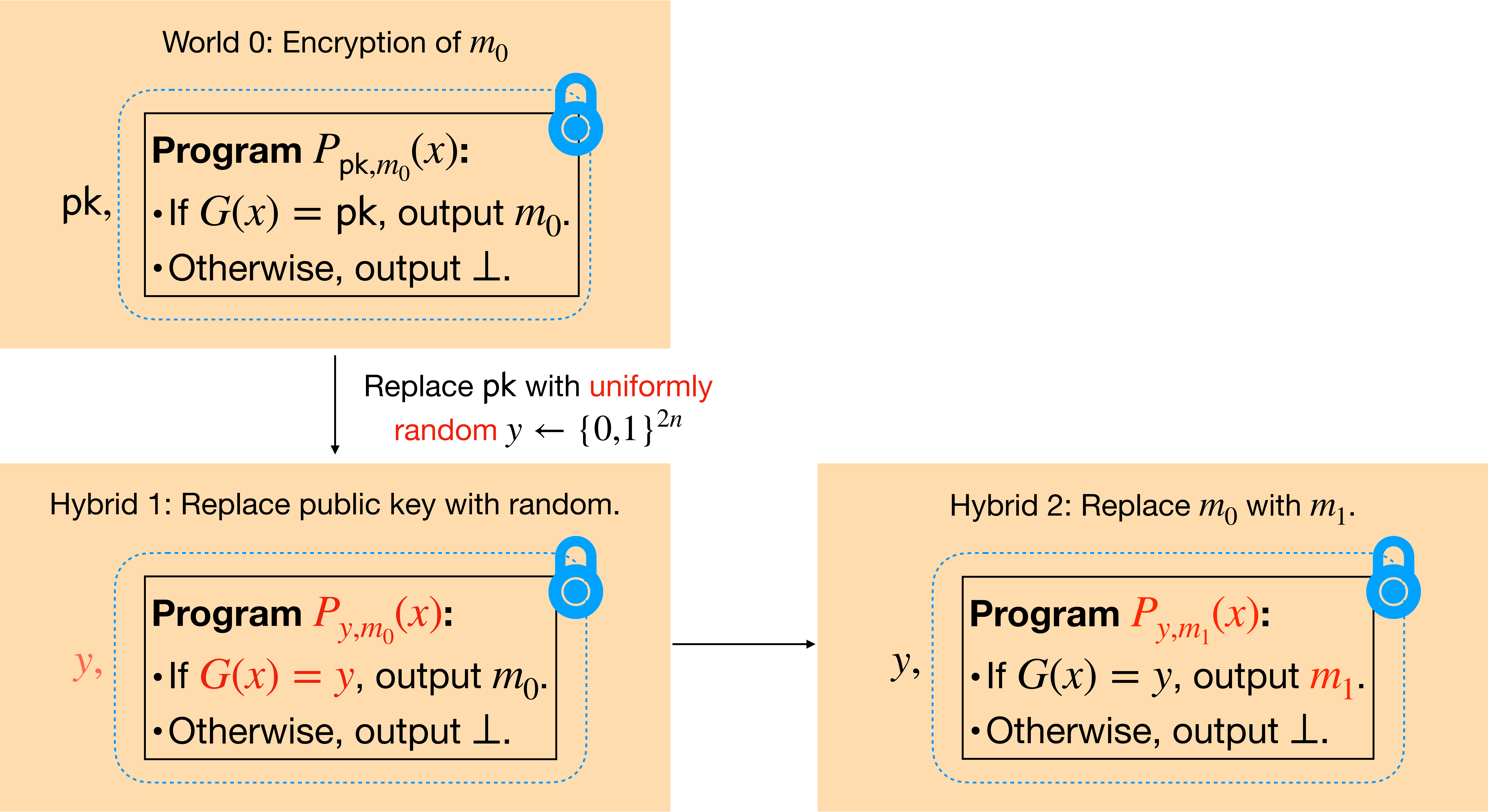
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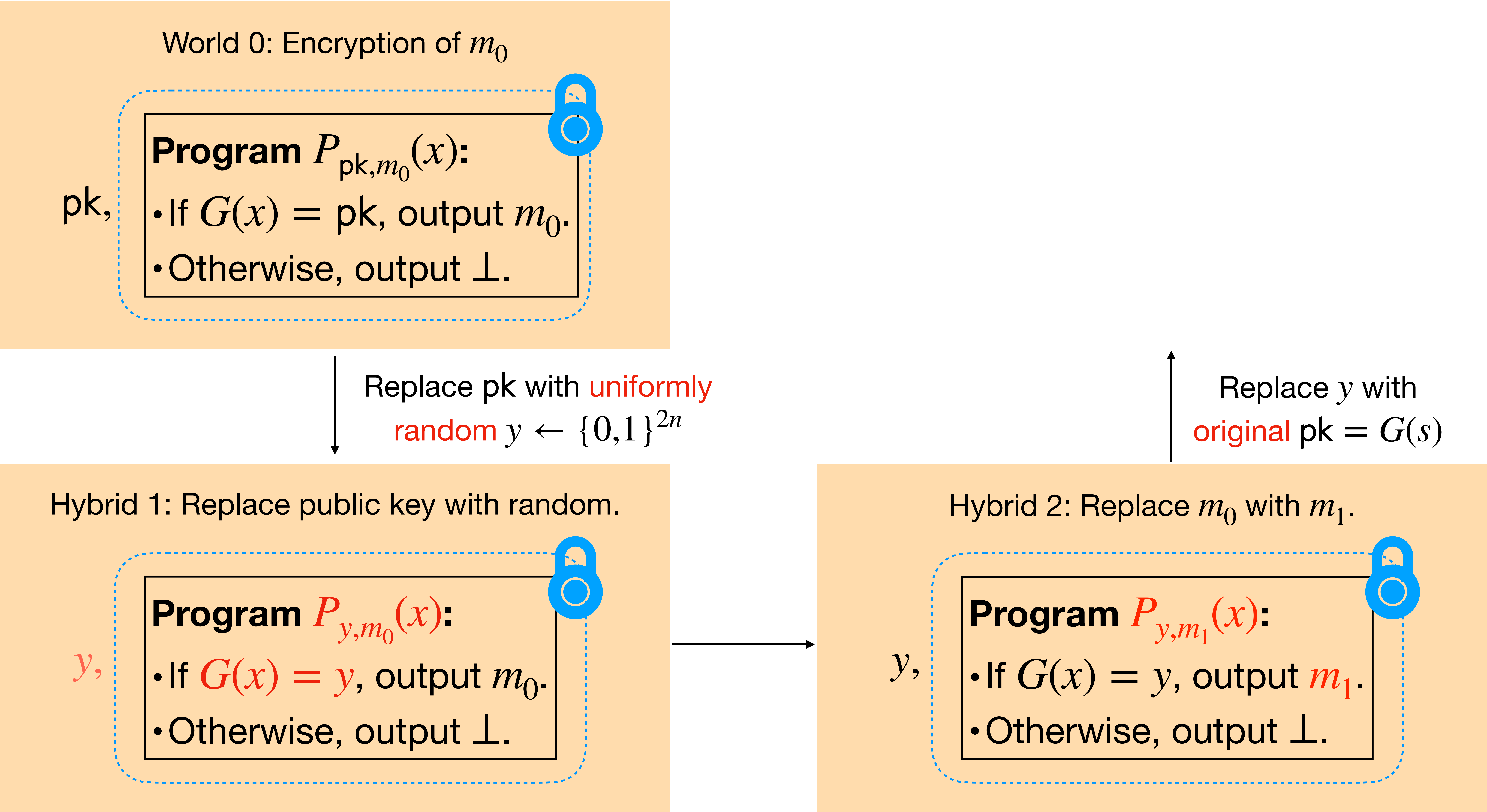

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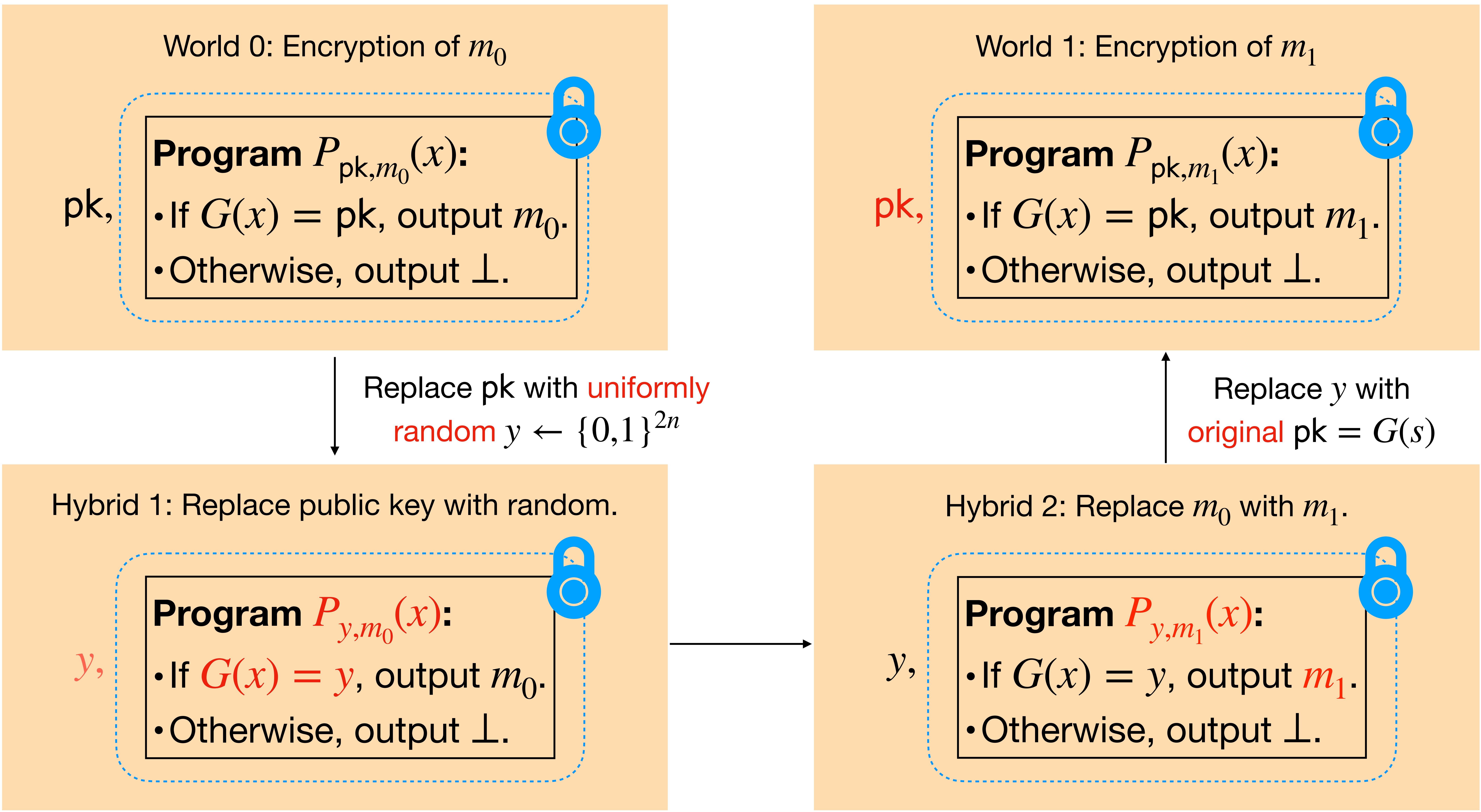
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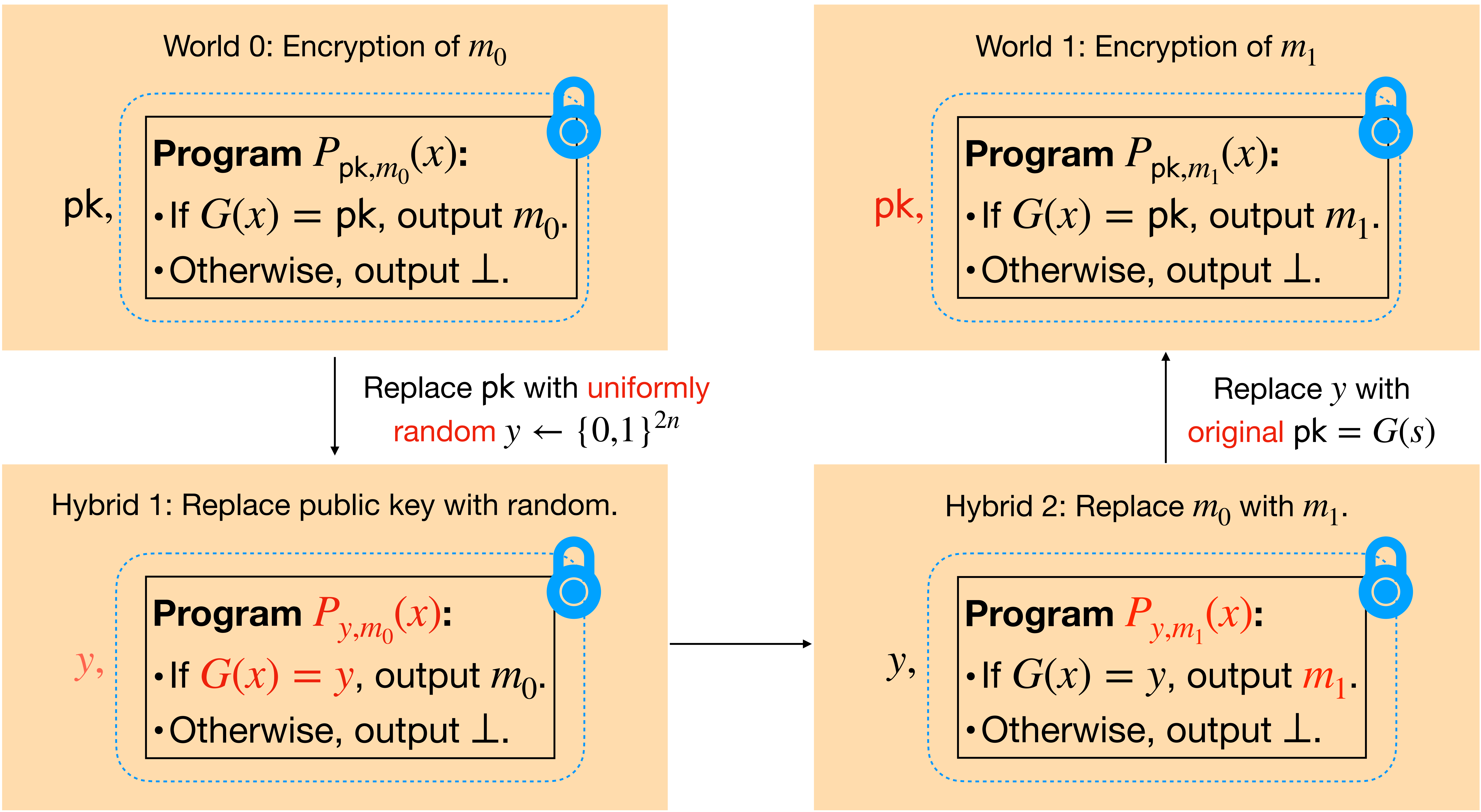
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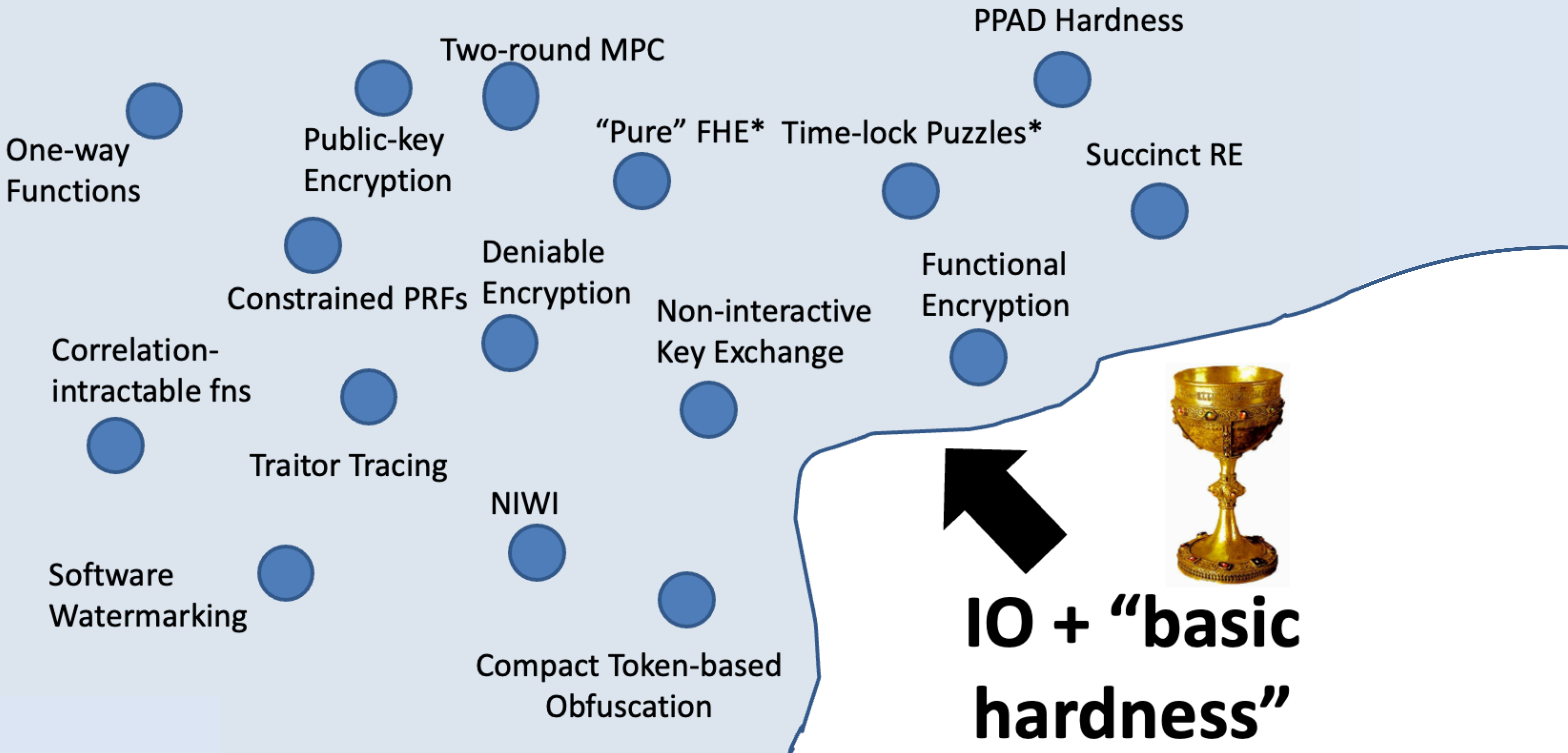
□

"CRYPTO-COMPLETE":

IO + Basic Hardness + Hard Work ⟹ Nearly all crypto.



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- **(Security)** For all $x \notin L$, $WEnc(x,0) \approx_c WEnc(x,1)$.

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Program $P_{x,m}(w)$**:**

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Theorem. iO gives us WE :)

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- WDec(w, c): Interpret c as a program, run it on input w, and output the result.

Can we construct an indistinguishability obfuscation scheme?

My Bounty

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Witness Encryption and Null-IO from Evasive LWE

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Hoeteck Wee† — NTT Research
Daniel Wichs‡ — Northeastern U. and NTT Research

August 31, 2022

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