MIT 6.875

Foundations of Cryptography
Lecture 25

Credit: Some slides adapted from Raluca Ada Popa and Vinod Vaikuntanathan

$O(1)$-Round Two-Party Computation

Secure Two-Party Computation

Input: x

Input: y

Alice

Bob

- Alice and Bob want to compute $F(x, y)$.

Semi-honest Security:

Parties should not learn anything more than their inputs and $F(x, y)$.

Secure Two-Party Computation

REAL WORLD:

Input: x

Input: y

Alice

Bob

$\approx$

IDEAL WORLD:

x

$F(x, y)$

y

$F(x, y)$

Secure Two-Party Computation

Input: x

Input: y

Alice

Bob

There exists a PPT simulator SIM_A such that for any x and y:

$$SIM_A(x, F(x, y)) \cong View_A(x, y)$$

Secure Two-Party Computation

Input: x

Input: y

Alice

Bob

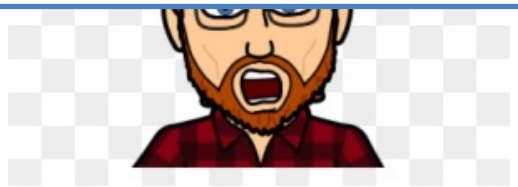
There exists a PPT simulator SIM_B such that for any x and y:

$$SIM_B(y, F(x,y)) \cong View_B(x,y)$$

Secure MPC

***Theorem* [Goldreich-Micali-Wigderson'87]:**
OT can solve ***any*** multi-party computation problem.

One year before '87…
Theorem (Yao'86):
OT**+OWF** solve ***any*** **two**-party computation problem.



Secure 2PC from OT

- **Constant Round!!**
- Groundbreaking generic solution
- Inspired GMW'87 and more
- **Beyond secure computation**
  - Computing on encrypted data
  - Secure function evaluation
  - Parallel cryptography
  - …
  - Garbling as **Randomized Encoding of functions** [IK'00,IK'02,AIK'04,AIK'06]…

Secure Function Evaluation

Alice's private input is $C: \{0,1\}^n \to x$
Bob's private input is x

Goal: Compute $C(x)$

Q: Is iO a solution?

A: depends…
❌ SFE needs interaction
✅ But SFE gives one-time evaluation *only*.

Secure 2PC from OT

- **Constant Round!!**
- Groundbreaking generic solution
- Inspired GMW'87 and more
- **Beyond secure computation**
  - Computing on encrypted data
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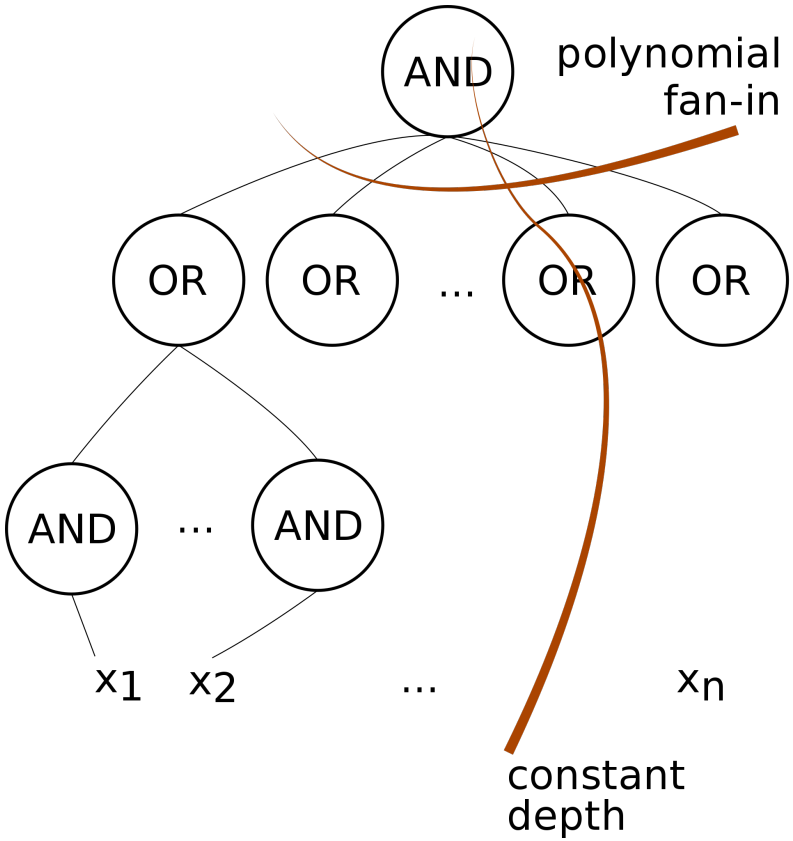
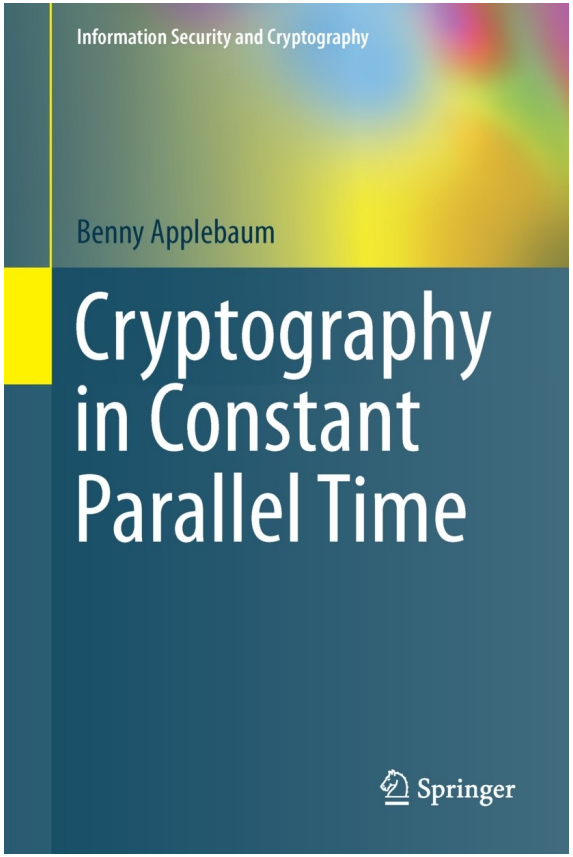
Parallel Cryptography

Can we do super-fast cryptography?





Complexity of the 2-party solution

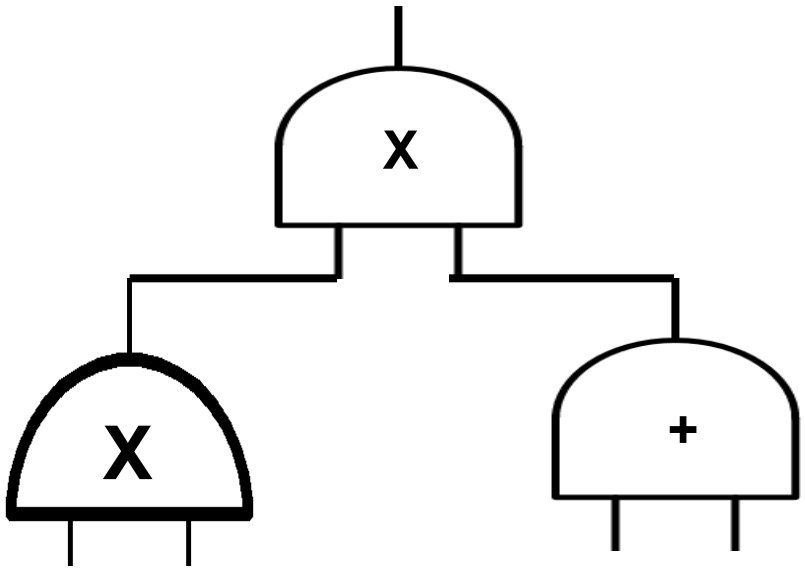
f computed by circuit C

$$C(x, y): \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^m$$

2PC efficiency	GMW'87	Garbled Circuit
# OT	$O(\#\wedge)$	$O(n \cdot \lambda)$
# Rounds	$\wedge$ -depth	$O(1)$
# Comm	$O(\#\wedge \cdot \lambda + m)$	$O(\lambda \cdot \#\wedge)$

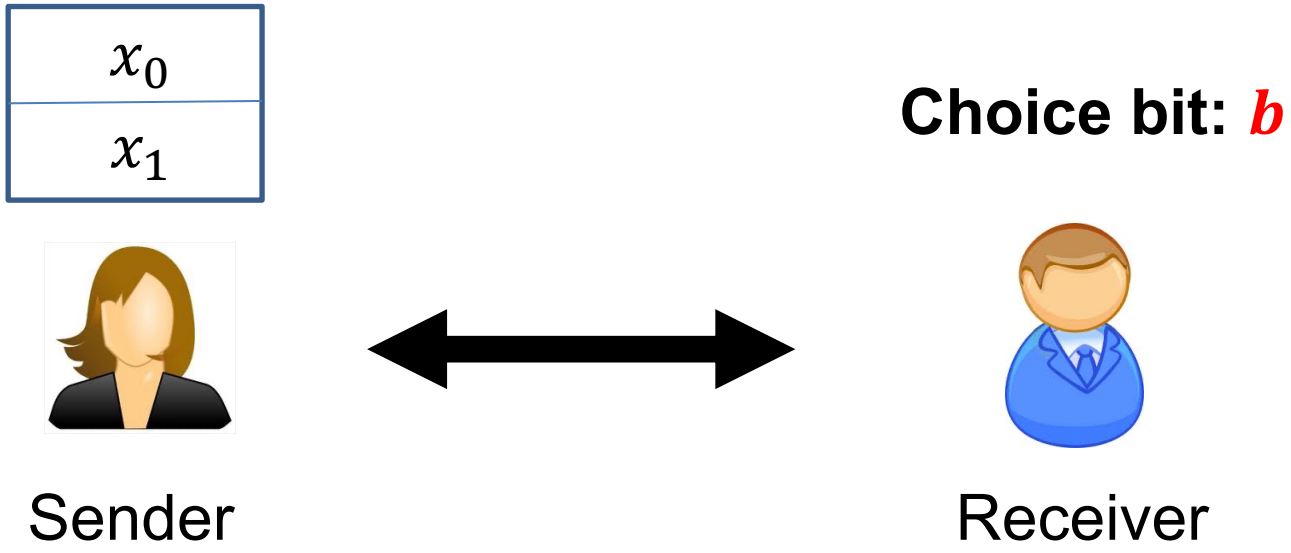
How to Compute Arbitrary Functions

For us, programs = functions = Boolean circuits with XOR ($+ \; mod \; 2$) and AND ($\times \; mod \; 2$) gates.



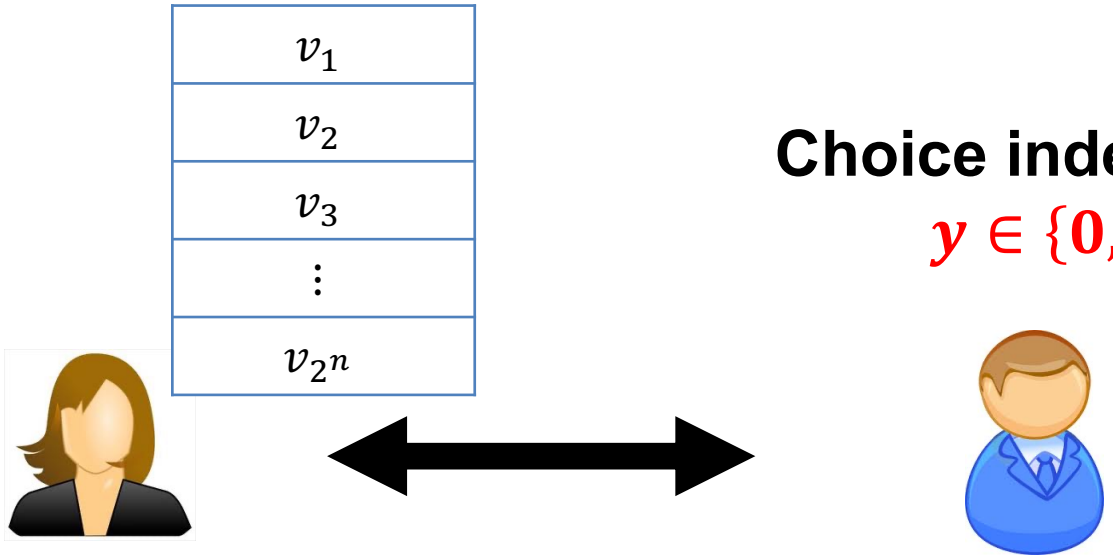
Goal: Compute every gate without knowing what the inputs and outputs are

Tool: Oblivious Transfer (OT)



- Sender holds two bits x_0 and x_1.
- Receiver holds a choice bit b.
- Receiver should learn x_b, sender should learn nothing.

What if we have 1-out-of-2^n OT?



= PIR except we also require that receiver learns nothing but x_y.

- Receiver should learn x_y, sender should learn nothing.

OT on Truth table?

Input: $x \in \{0,1\}^n$

$f(x, 0^n)$
$f(x, 0^{n-1}1)$
$f(x, 0^{n-2}10)$
⋮
$f(x, 1^n)$

Input: $y \in \{0,1\}^n$

Alice

Bob

- itself is already inefficient!

Gate-by-gate on circuit of f !

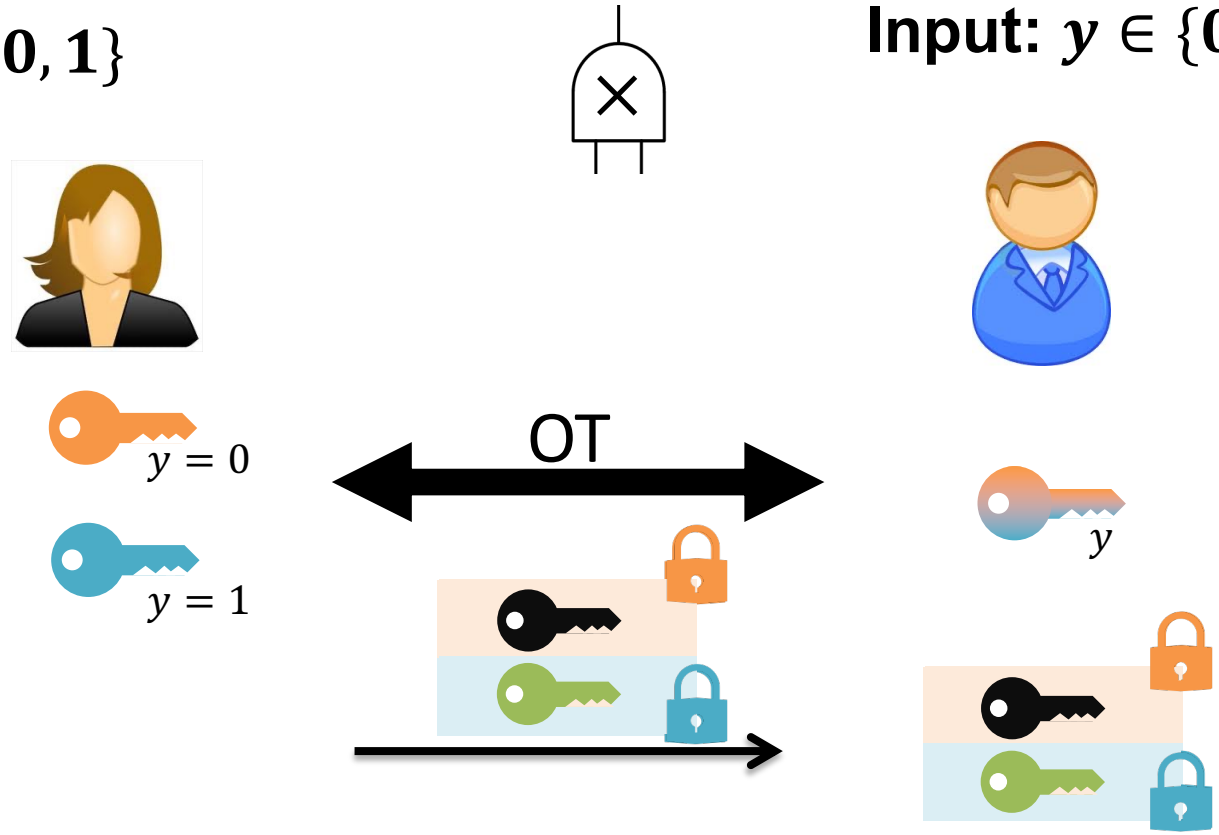
Instead of deriving the OT-by-OT protocol of GMW'87…

Fun with Lockboxes

Physical 2PC

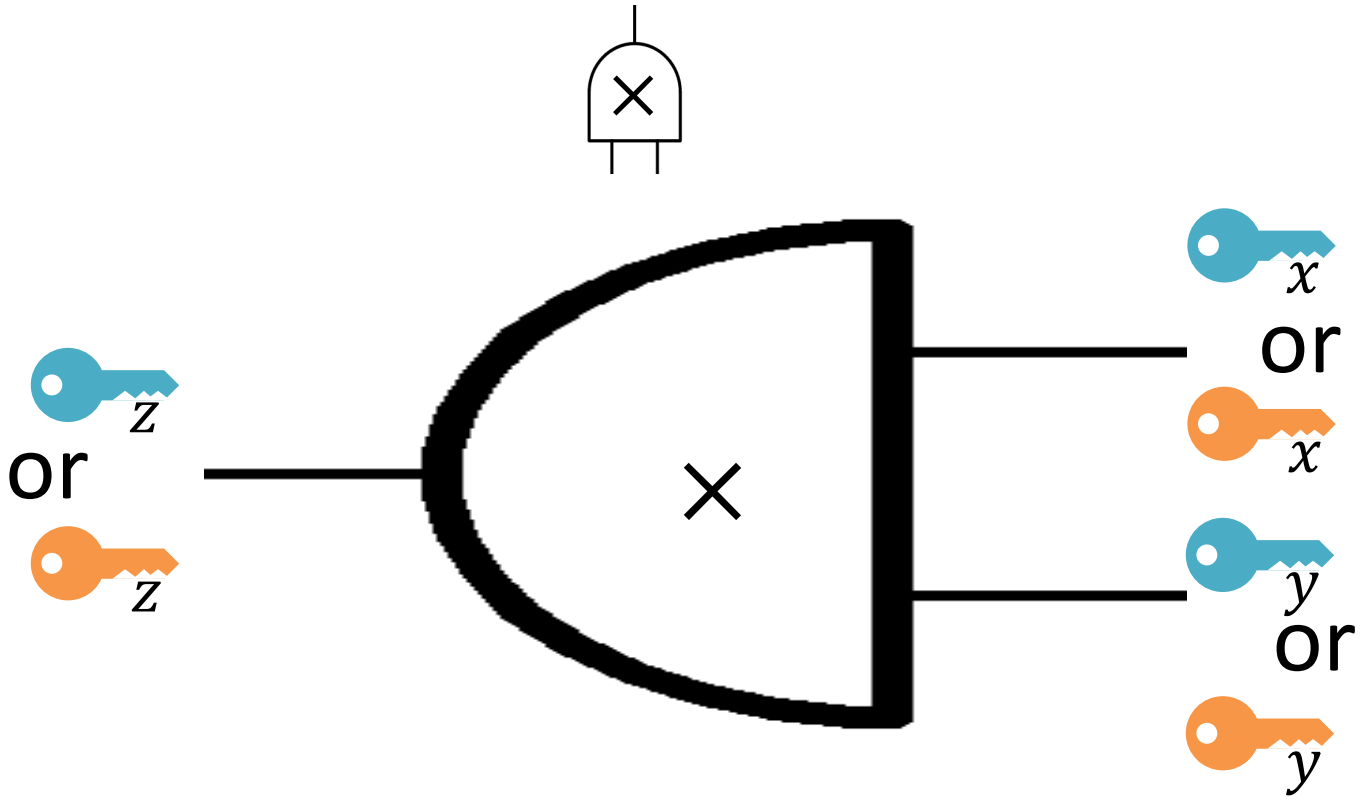
Input: $x \in \{0,1\}$

Input: $y \in \{0,1\}$



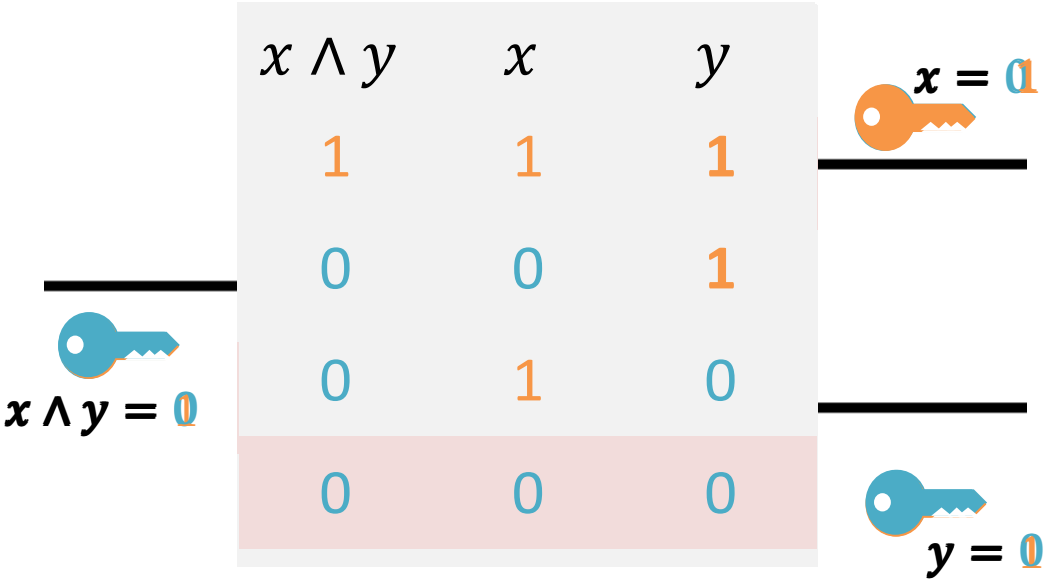
Possession of or is information!

Physical 2PC

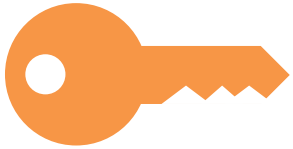


- Blue/orange means 1/0, but keys on different wires, even with same colors, are different

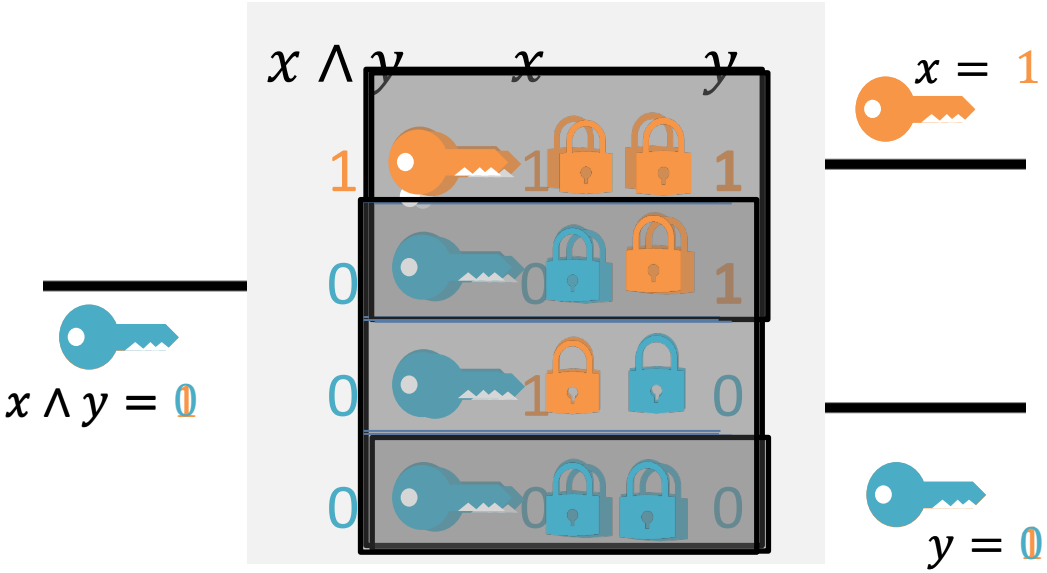
Idea



Key Invariant: For each wire w of the circuit, generate a pair of keys k_w^0, k_w^1. The possession of $k_w^b \Leftrightarrow$ the value carried on the wire is b.

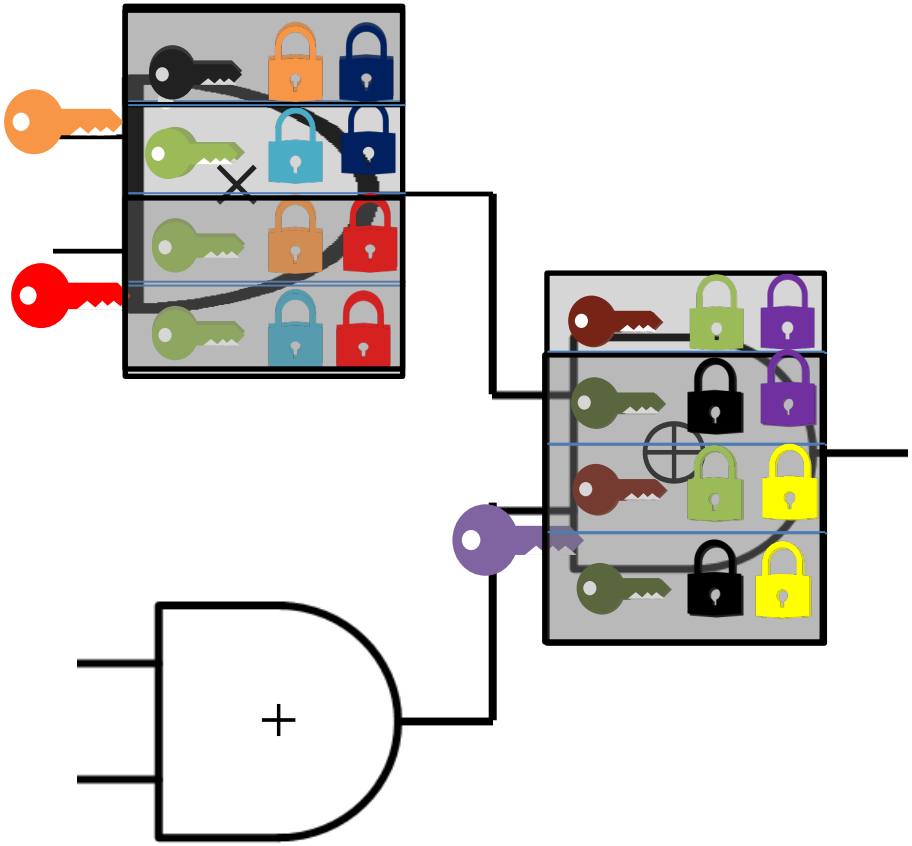


Idea



Key Invariant: For each wire w of the circuit, generate a pair of keys k_w^0, k_w^1. The possession of $k_w^b \Leftrightarrow$ the bit b is carried on w.

Garbled Evaluation

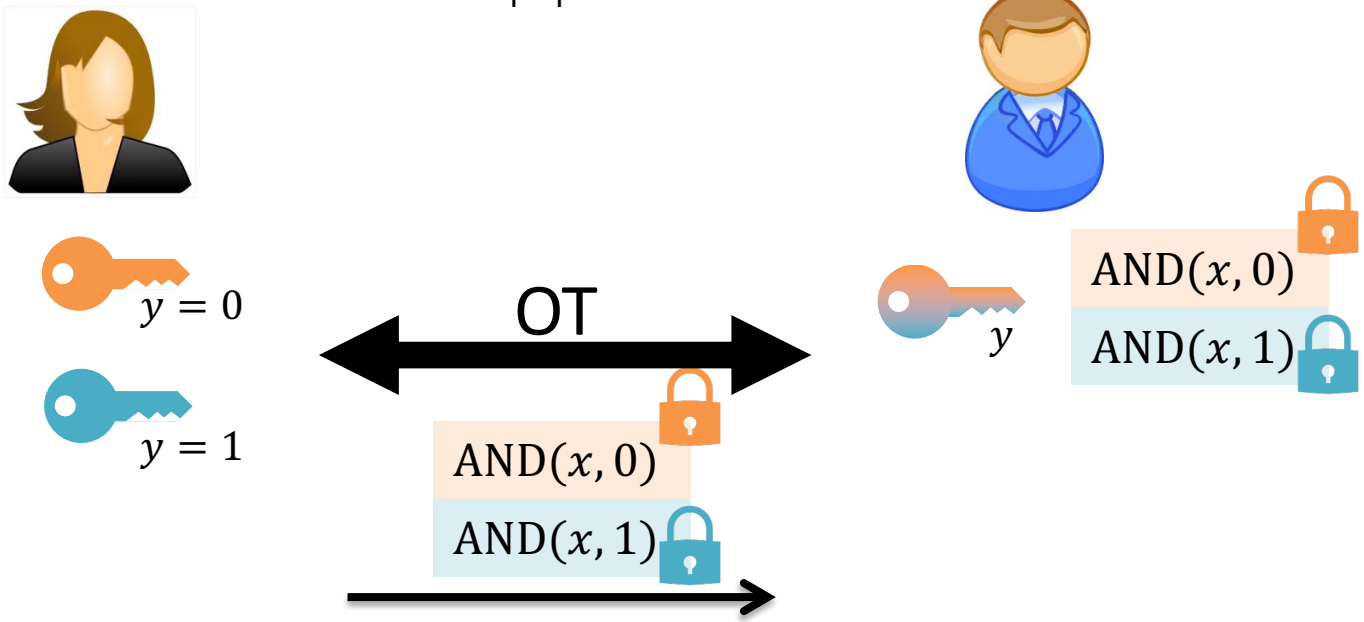


Crypto Lockboxes

Crypto 2PC

Input: $x \in \{0,1\}^n$

Input: $y \in \{0,1\}\%$



- Alice puts results in IND-CPA encryption
- Bob gets key via some crypto OT mechanism
- Bob tries to open both boxes using the key he received.

What about last point?

Tool: *Special* CPA Encryption

CPA-secure secret-key encryption $(\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ that satisfies

1. **Elusive range** --- Let $k \leftarrow_R \mathrm{Gen}(1^n)$. Any p.p.t. adversary A cannot generate ciphertext encrypted under k, w/o k.
2. **Efficiently verifiable range** --- there exists an algo $\mathrm{Check}(1^n, k, c)$ that checks if c is encrypted under k.

Tool: *Special* CPA Encryption

1. **Elusive range** --- Let $k \leftarrow_R \mathrm{Gen}(1^n)$. Any p.p.t. adversary A cannot generate ciphertext encrypted under k, w/o k.
2. **Efficiently verifiable range** --- there exists an algo $\mathrm{Check}(1^n, k, c)$ that checks if c is encrypted under k.

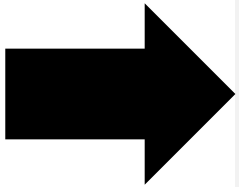
Construction:

Let $F_k: \{0,1\}^n \rightarrow \{0,1\}^{2n}$ be a PRF,

$$\mathrm{E}_k(m; r) := F_k(r) \oplus (m || 0^n).$$

Crypto Lockboxes

x	y	z
1	1	1
0	1	0
1	0	0
0	0	0



Garbled gate

$$\mathrm{E}_{k_x^1}(\mathrm{E}_{k_y^1}(k_z^1))$$

$$\mathrm{E}_{k_x^0}(\mathrm{E}_{k_y^1}(k_z^0))$$

$$\mathrm{E}_{k_x^1}(\mathrm{E}_{k_y^0}(k_z^0))$$

$$\mathrm{E}_{k_x^0}(\mathrm{E}_{k_y^0}(k_z^0))$$

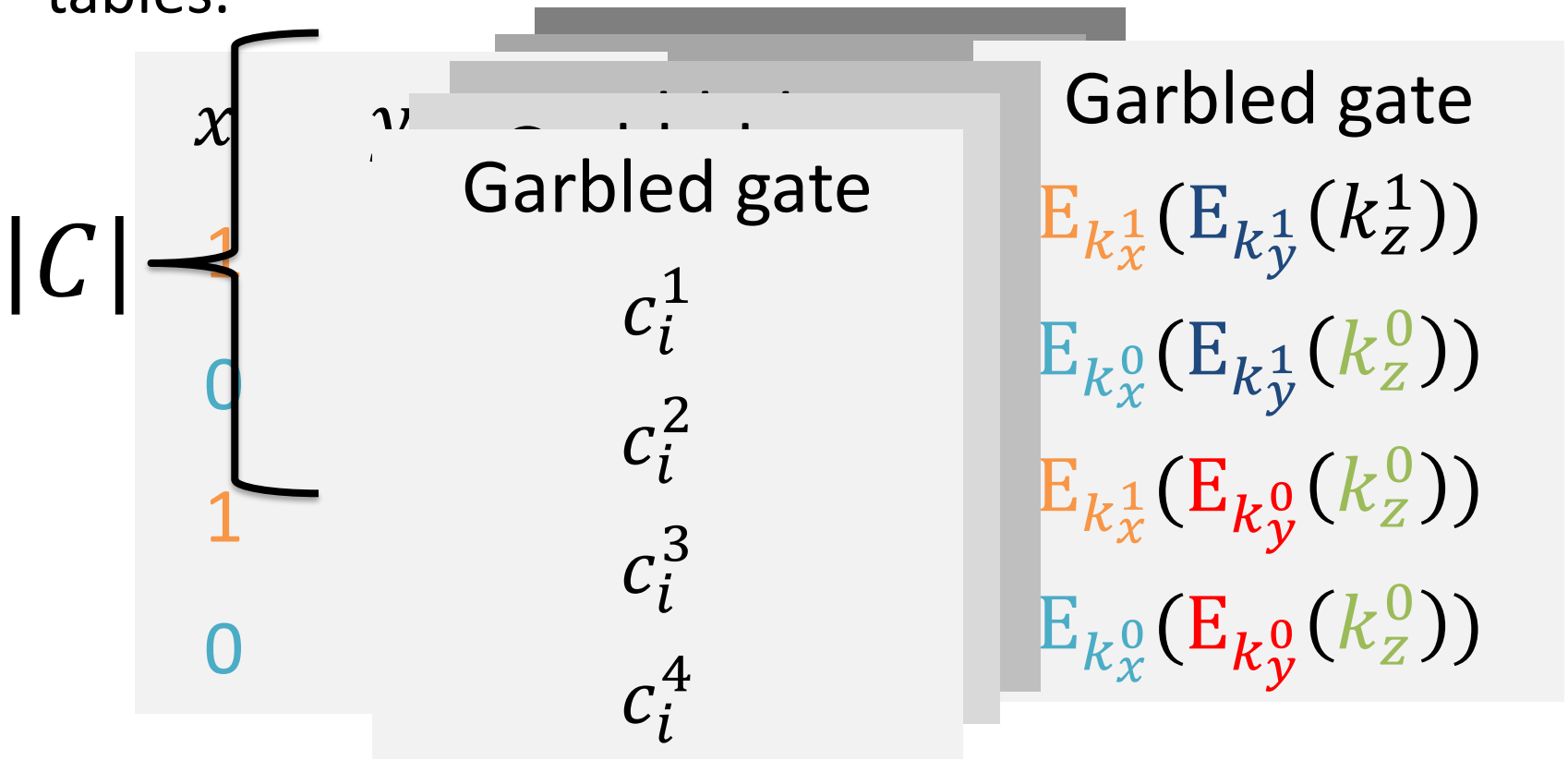
Efficiently verifiable range

$\mathrm{Check}(1^n, k, c)$ checks if c is encrypted under k.

Protocol Sketch

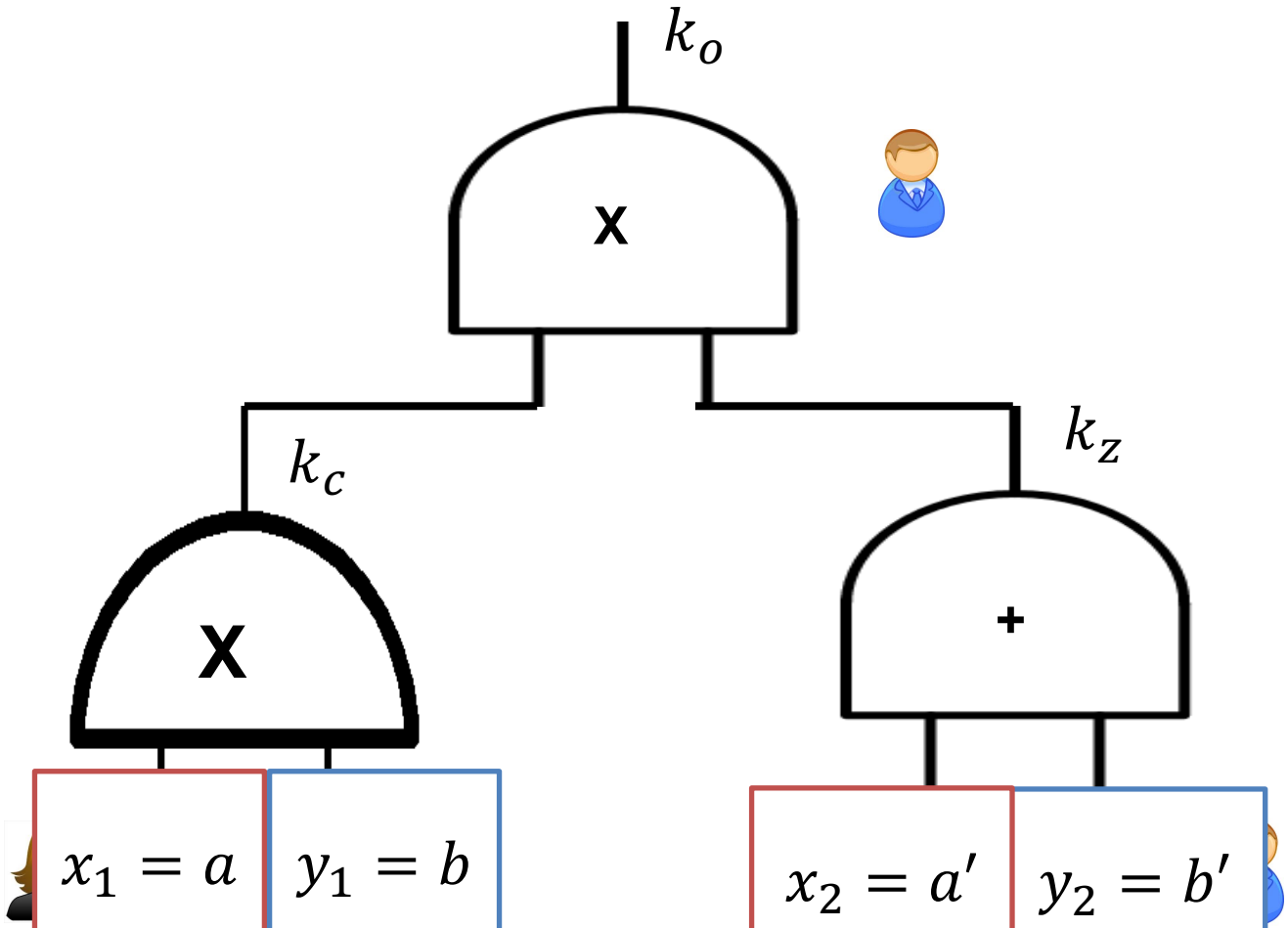
Yao's protocol: Garbling

- *Keys generation:* For each wire w of C, Alice generates a pair of keys k_w^0, k_w^1.
- *Gate Garbling:* For each gate $z \leftarrow G(x, y)$, compute the tables.



Yao's protocol: Evaluator Bob

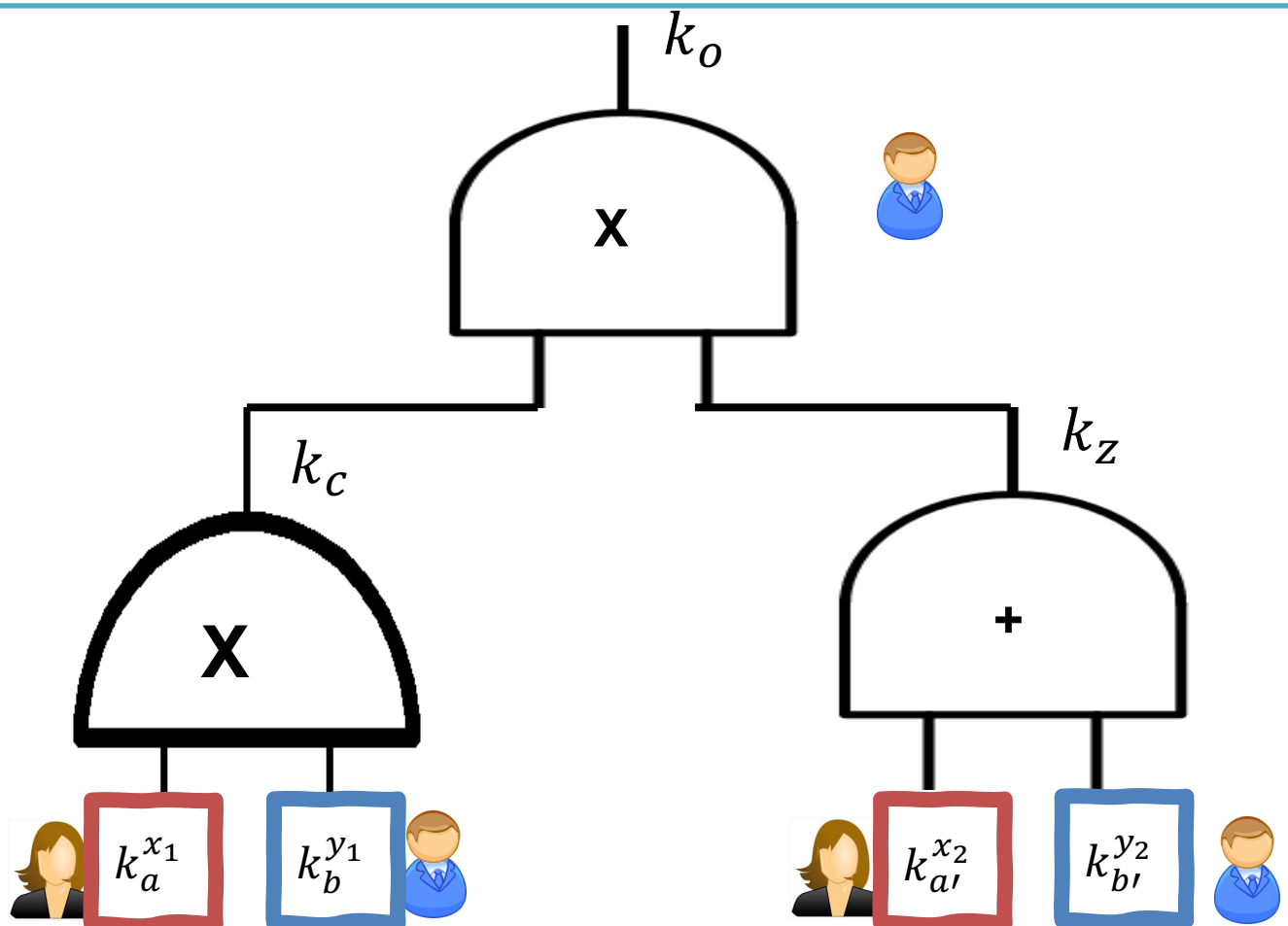
Key Invariant: For each wire w of the circuit, Bob can obtain exactly one of the two keys from k_w^0, k_w^1.



Yao's protocol: Evaluator Bob

Base case:

- Alice's input $x_i \in \{0,1\}$: Alice send the correct key $k_i^{x_i}$.
- Bob's input $y_i \in \{0,1\}$: they runs OT on $((k_i^0, k_i^1), y_i)$.



Yao's protocol: Evaluator Bob

Inductive step:

Assume: Bob has one key for each input wire

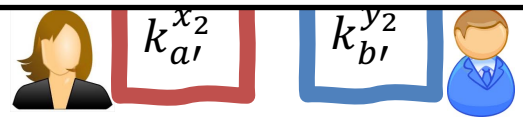
- Bob can get exactly one key for the output wire by trying all four ciph

$k_o^{(x_1+x_2)(x_1 \wedge x_2)}$

Efficiently verifiable range

$\mathrm{Check}(1^n, k, c) \in \{0,1\}$ checks if

c is encrypted under k.



Evaluating One Gate

Inductive step:

Oops..
This procedure, as-is, is actually insecure.

trying all four ciphertexts.

Recall:

Given $k_x^{b_x}, k_y^{b_y}$,

- Try all four rows to obtain $k_z^{b_x \wedge b_y}$

Garbled gate

$$\mathrm{E}_{k_x^1}(\mathrm{E}_{k_y^1}(k_z^1))$$

$$\mathrm{E}_{k_x^0}(\mathrm{E}_{k_y^1}(k_z^0))$$

$$\mathrm{E}_{k_x^1}(\mathrm{E}_{k_y^0}(k_z^0))$$

$$\mathrm{E}_{k_x^0}(\mathrm{E}_{k_y^0}(k_z^0))$$

Reconstructing Output

Key Invariant: For each wire of the circuit, Bob can obtain exactly one of the two keys associated with each wire.

After evaluation, Bob learns $k_o \in \{k_o^0, k_o^1\}$.

- Bob simply asks Alice if k_o is k_o^0 or k_o^1.

$k_o^{(x_1+x_2)(x_1 \wedge x_2)}$

Can we avoid this final round?

Garbling as a Standalone Tool

Q: Difference with $i\mathcal{O}$?

- **Input:** Boolean circuit $C: \{0,1\}^n \to \{0,1\}$
- **Output:** Garbled circuit $G(C)$ and input labels $\{(L_1^0, L_1^1), \ldots, (L_n^0, L_n^1)\}$

Goal: Given $G(C)$ and $L_1^{x_1}, \ldots, L_n^{x_n}$

- It is possible to compute $C(x_1 \cdots x_n)$
- It is not possible to learn any additional information other than size of circuit or input

For example, for $x = 010$, labels are L_1^0, L_2^1, L_3^0

2PC Using Garbled Circuits

Input will be x, y

Common input: $C: \{0,1\}^{2n} \rightarrow \{0,1\}$

Input: $x \in \{0,1\}^n$

Compute $G(C)$ and labels $\{(L_i^0, L_i^1)\}_{i \in [2n]}$

Garbled circuit $G(C)$ →

Input labels $L_1^{x_1}, \ldots, L_n^{x_n}$ for x →

OT for each $i \in [n]$ in parallel:

- Alice's input: (L_{n+i}^0, L_{n+i}^1)
- Bob's input: y_i

← $C(x, y)$

Input: $y \in \{0,1\}^n$

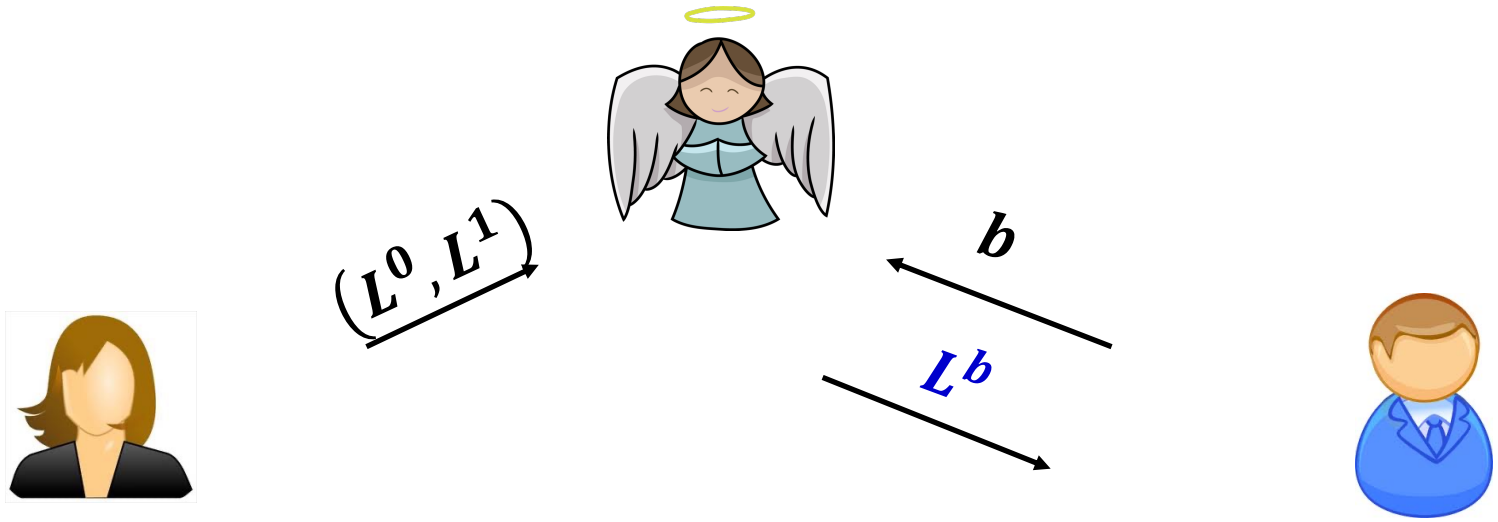
Compute $C(x, y)$ using $G(C)$ and $L_1^{x_1}, \ldots, L_n^{x_n}, L_{n+1}^{y_1}, \ldots, L_{2n}^{y_n}$

Simulation Proof Sketch

Simulating Alice

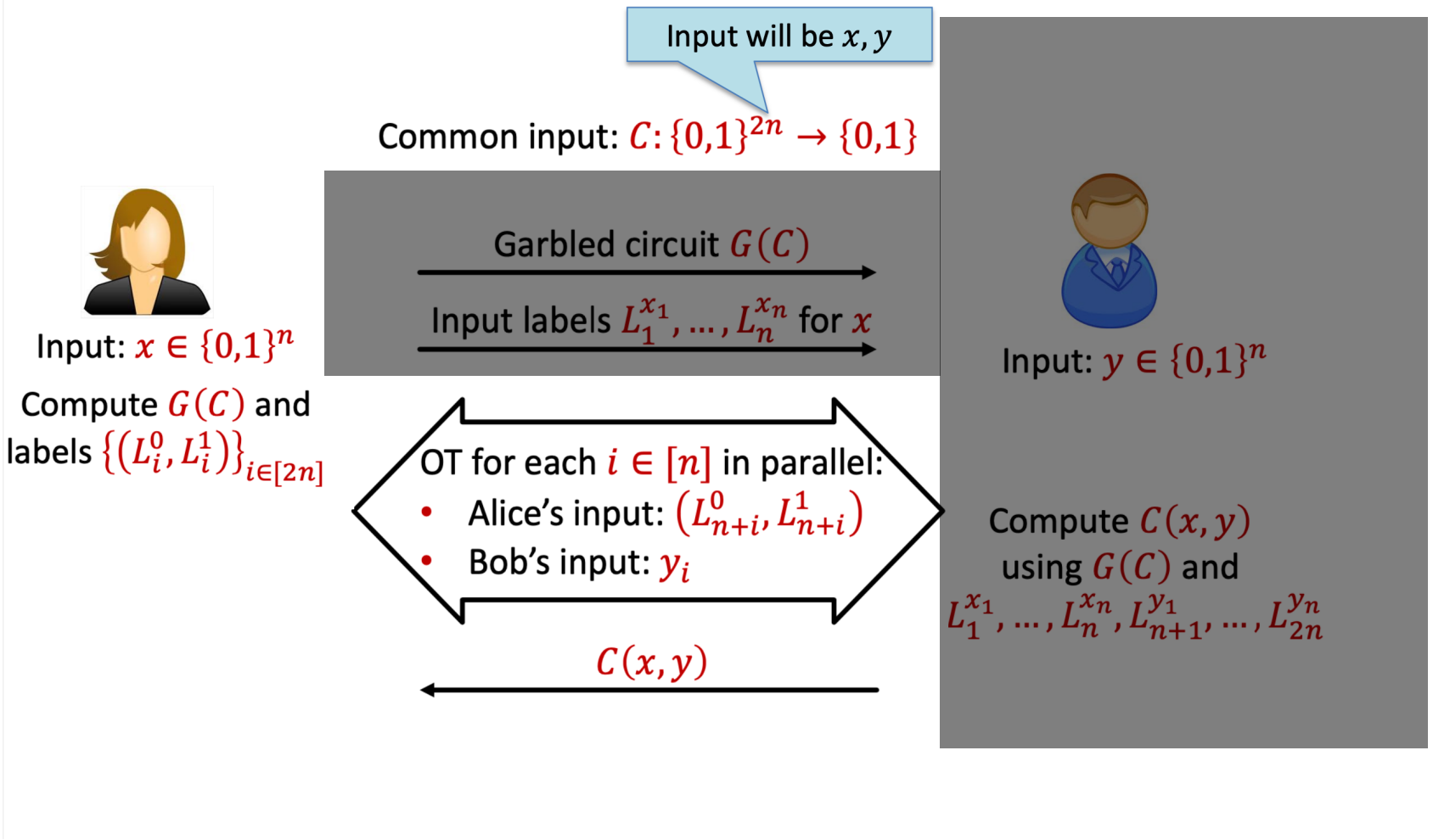
Imagine that the parties have access to an OT angel

- Implemented by secure simulatable OT.



Simulating Alice

Imagine that the parties have access to an OT angel.



Simulating Alice

Alice's View

- OT transcripts
- $C(x, y)$

$\approx$

$\mathrm{Sim}\big(C, x, f(x, y)\big)$:

Output

$\{\{\mathrm{Sim}_{\mathrm{OT}}^{A}\big(L_{n+i}^{0}, L_{n+i}^{1}\big)\}_i$

$f(x, y)\}$

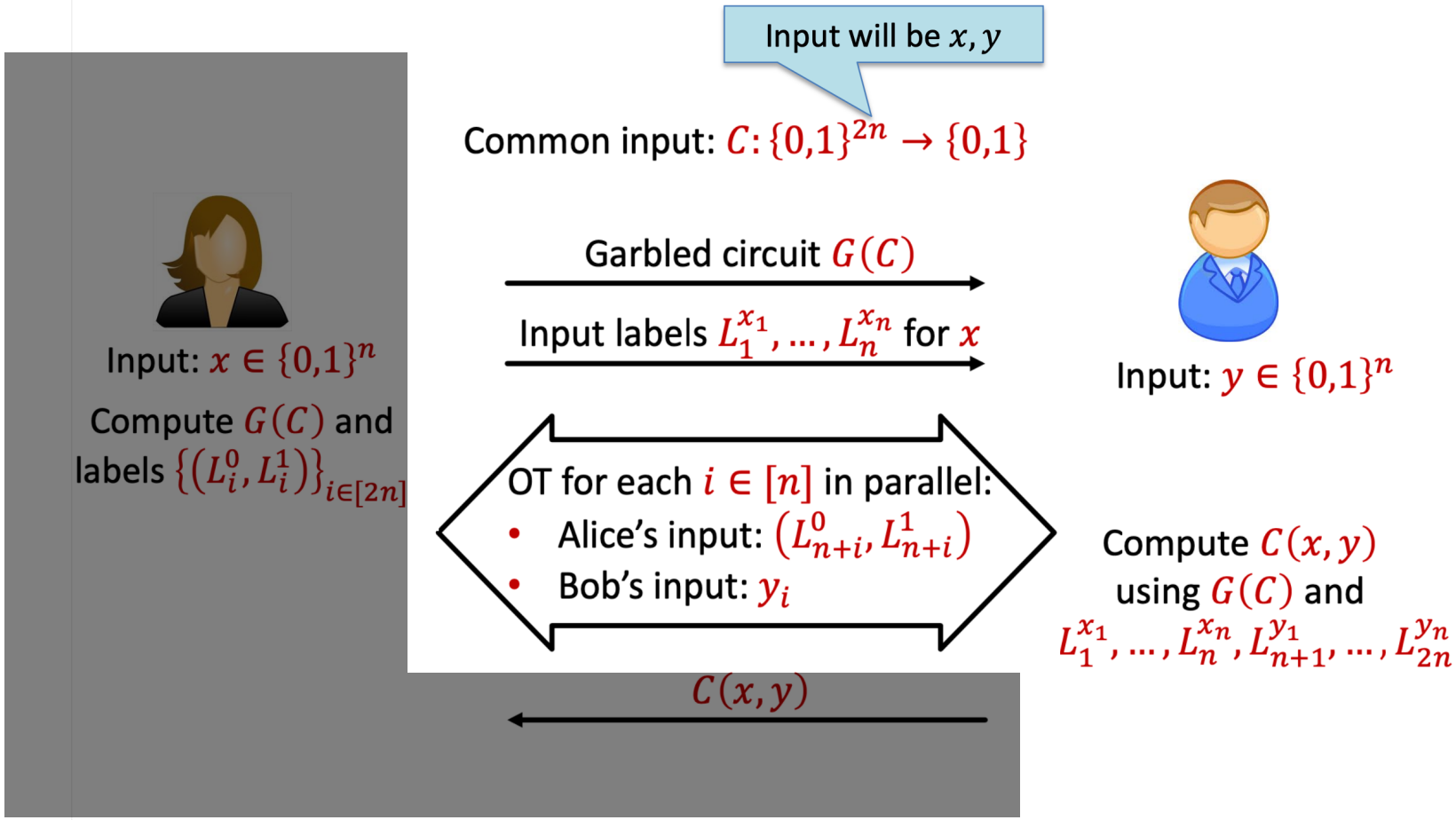
OT for each $i \in [n]$ in parallel:

- Alice's input: $\big(L_{n+i}^{0}, L_{n+i}^{1}\big)$
- Bob's input: y_i

$C(x, y)$

Simulating Bob

Imagine that the parties have access to an OT angel.



Simulating Bob

OT Simulation:

$$\mathrm{View}_{\mathrm{OT}}^{\mathrm{B}} \approx \mathrm{Sim}_{\mathrm{OT}}^{\mathrm{B}}\left(1^n, y_i, L_{n+i}^{y_i}\right).$$

OT for each $i \in [n]$ in parallel:

- Alice's input: $\left(L_{n+i}^0, L_{n+i}^1\right)$
- Bob's input: y_i

Assume we already simulated L_{n+i}^B with the correct distribution.

Simulating Bob

Input will be x, y

Common input: $C: \{0,1\}^{2n} \to \{0,1\}$

Input: $x \in \{0,1\}^n$

Compute $G(C)$ and labels $\{(L_i^0, L_i^1)\}_{i \in [2n]}$

Garbled circuit $G(C)$

Input labels $L_1^{x_1}, \ldots, L_n^{x_n}$ for x

Input: $y \in \{0,1\}^n$

OT for each $i \in [n]$ in parallel:

- Alice's input: (L_{n+i}^0, L_{n+i}^1)
- Bob's input: y_i

Compute $C(x, y)$ using $G(C)$ and $L_1^{x_1}, \ldots, L_n^{x_n}, L_{n+1}^{y_1}, \ldots, L_{2n}^{y_n}$

$C(x, y)$

Simulating Bob

Garbled circuit $G(C)$

Input labels $L_1^{x_1}, \ldots, L_n^{x_n}$ for x

Input: $y \in \{0,1\}^n$

Step 1: Generate Dummy Labels

- *Label generation:* For each wire w of C, generates a pair of keys L_w^0, L_w^1.
- *Label simulation:* For all input wire i, let $\widetilde{L_i^{x_i}} := L_i^0$.

Sanity Check:
This replacement is fine because keys are randomly generated.

Step 2.a: Simulate Fake Gates

- *Garbled gate simulation:* Replace intermediate ciphertexts with junks.

$\sigma \cdot$

Garbled gate

$\mathrm{E}_{L_x^1}(\mathrm{E}_{L_y^1}(L_z^1))$

$\mathrm{E}_{L_x^0}(\mathrm{E}_{L_y^1}(L_z^0))$

$\mathrm{E}_{L_x^1}(\mathrm{E}_{L_y^0}(L_z^0))$

$\mathrm{E}_{L_x^0}(\mathrm{E}_{L_y^0}(L_z^0))$

$\approx$

$\tau \cdot$

Garbled gate

$\mathrm{E}_{L_x^1}(\mathrm{E}_{L_y^1}(L_z^0))$

$\mathrm{E}_{L_x^0}(\mathrm{E}_{L_y^1}(L_z^0))$

$\mathrm{E}_{L_x^1}(\mathrm{E}_{L_y^0}(L_z^0))$

$\mathrm{E}_{L_x^0}(\mathrm{E}_{L_y^0}(L_z^0))$

- the rows are randomly permuted ($\sigma, \tau \in Perm([4])$)
- only a random row can be decrypted
- the junk entries are w.h.p. non-decryptable.

Step 2.b: Simulate Output

- *Generate the following decoding table*

Output label	Decoded result
L_o^0	$C(x, y)$
L_o^1	$1 - C(x, y)$

Sanity Check: This is fine because the label L_o^0 might as well be encoding 1.

Input: $y \in \{0,1\}^n$

Simulating Bob

Bob's View

- Wire labels
- Garbled tables
- Final decoding table
- OT transcripts

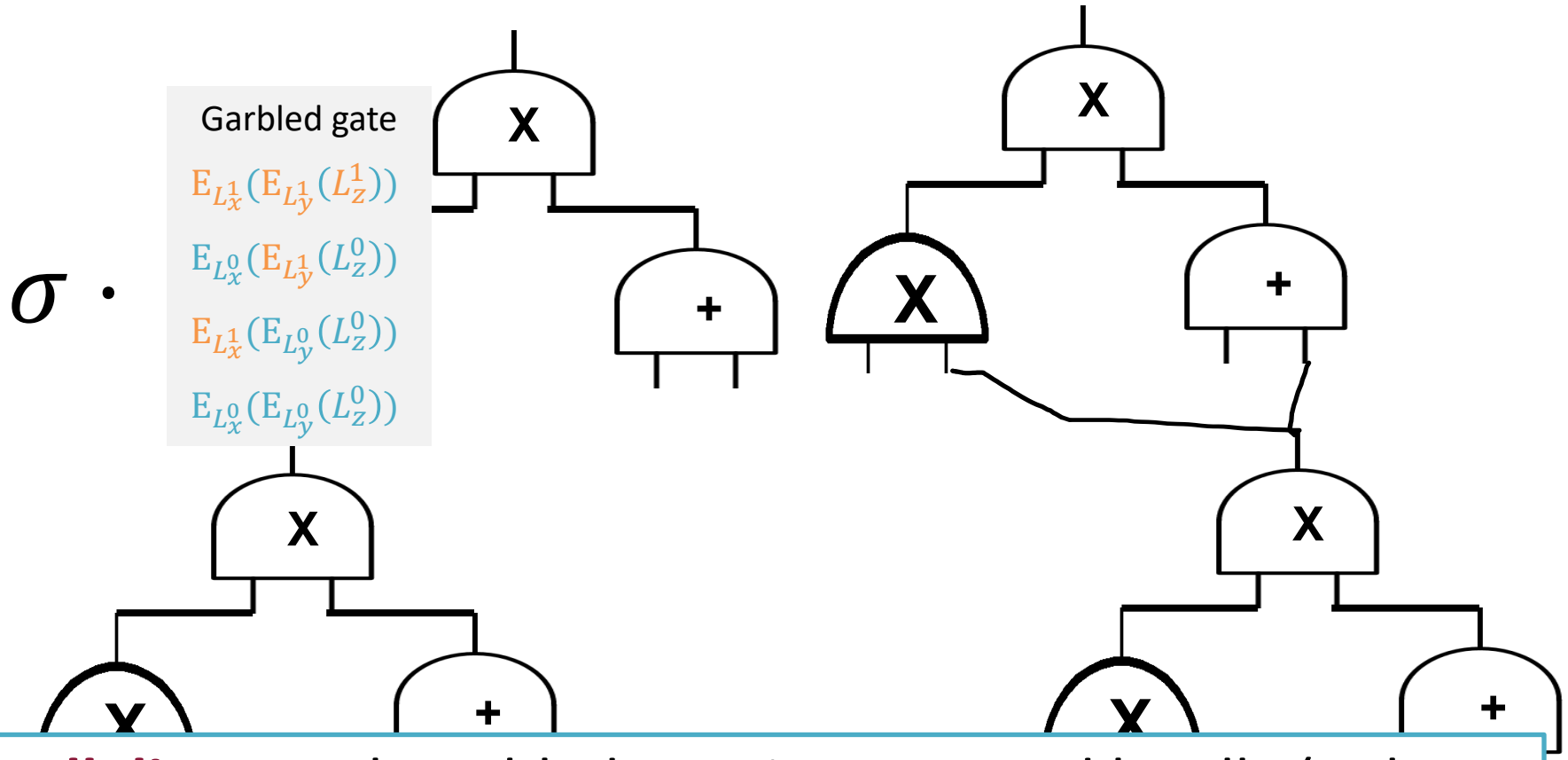
$\approx$

$\mathrm{Sim}\left(C, y, f(x,y)\right)$:

- Simulate labels
- $\mathrm{Sim}_{\mathrm{OT}}^{\mathrm{B}}\left(L_y^{y_i}\right)$ 
- Simulate garbled gates
- Simulate final decoding table.

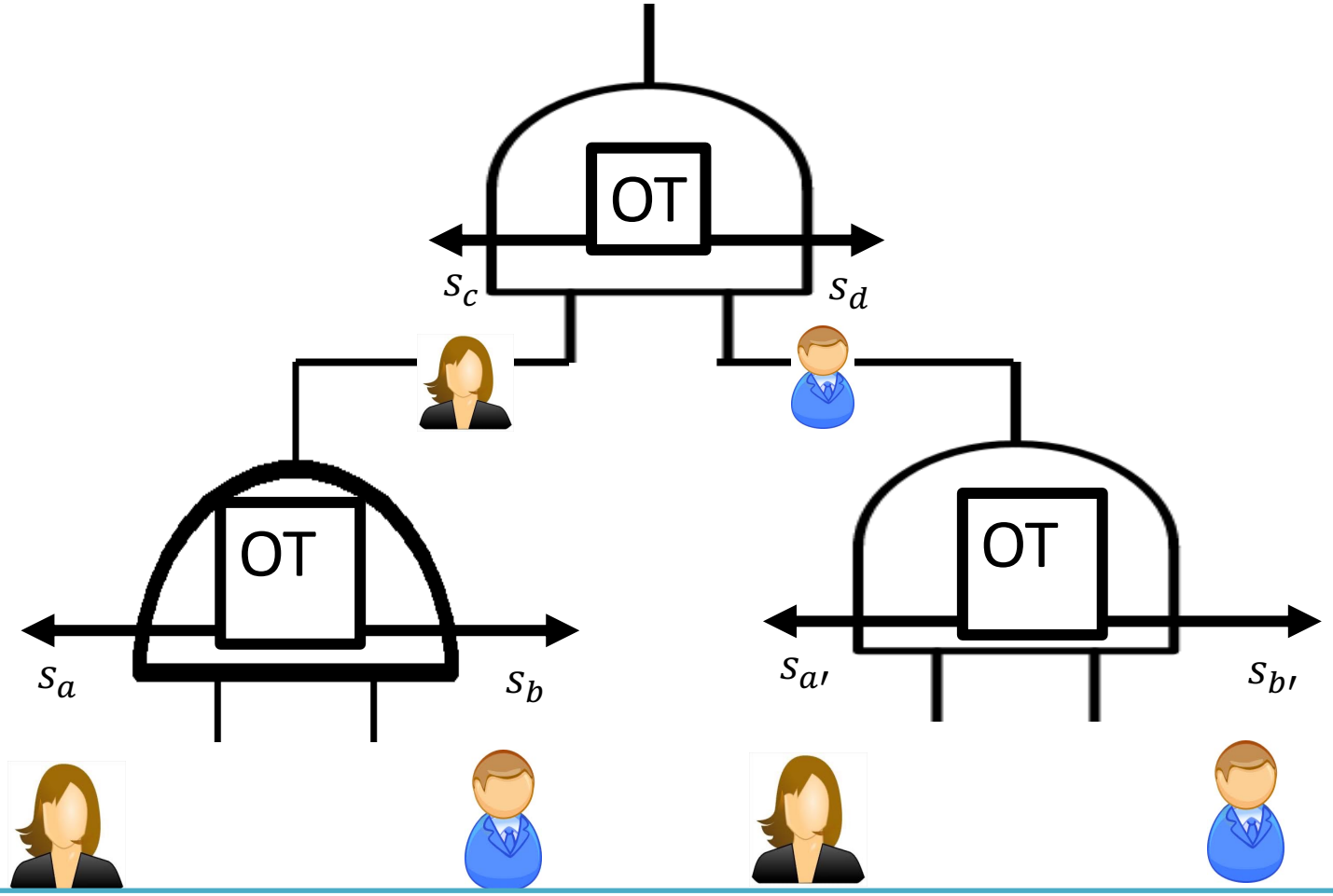
Efficiency

Garbling is parallelizable



Parallelism: Each garbled gate is computed locally (only depends on $(L_x^0, L_x^1), (L_y^0, L_y^1), (L_z^0, L_z^1)$, generated at the very beginning).

Why is GMW sequential?



Sequentiality: Input to next OT is output from previous OT.

Garbled-circuit 2PC

f computed by circuit C

$$C(x, y): \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^m$$

2PC efficiency	GMW'87	Garbled Circuit
# OT	$O(\#\wedge)$	$O(n \cdot \lambda)$
# Rounds	$\wedge$ -depth	$O(1)$
# Comm	$O(\#\wedge \cdot \lambda + m)$	$O(\lambda \cdot \#\wedge)$

Optimization 1: Free XOR trick

In GMW or BGW, linear (XOR) gates are free (no communication).
Can we say something for Garbled circuits?

Theorem [Kolesnikov, Schneider'08]: If we generate labels *carefully,* then there is no need to send garbled XOR tables.

Observation: do not use so much randomness.

Optimization 1: Free XOR trick

Rough intuition:

Acceptable correlations of labels:

- Pick global R, a random value hidden from evaluator
- Generate non-XOR-output wire w subject to

$$L_w^b = L_w^{1-b} \oplus R$$

Now if $z = x \oplus y$, we define $L_z = L_x \oplus L_y$.

1. $L_z^1 = L_x^0 \oplus L_y^1 = L_x^0 \oplus L_y^0 \oplus R = L_x^1 \oplus L_y^0$.
2. $L^0 = L^1 \oplus L^1 = L^0 \oplus L^0$

Observation: do not use so much randomness.

Optimization 2: Half-Gates

Half-gate trick [Zahur, Rosulek, Evans'15]: Keeping XOR gates free, AND gate can be 2 ct each.

	size per gate		calls to H per gate			
			generator		evaluator	
technique	XOR	AND	XOR	AND	XOR	AND
classical [31]	4	4	4	4	4	4
point-permute [3]	4	4	4	4	1	1
row reduction (GRR3) [27]	3	3	4	4	1	1
row reduction (GRR2) [28]	2	2	4	4	1	1
free XOR + GRR3 [20]	0	3	0	4	0	1
fleXOR [19]	$\{0,1,2\}$	2	$\{0,2,4\}$	4	$\{0,1,2\}$	1
half gates [this work]	0	2	0	4	0	2

Table 1. Optimizations of garbled circuits. Size is number of "ciphertexts" (multiples of k bits).

Credit: Table taken from proceedings version of [ZRE'15].

Optimization 3: Beyond Half-Gates

Slicing & Dicing [Rosulek, Roy'21]: Keeping XOR gates free, AND gate can be 1.5 ct plus 5 bits each.

scheme	GC size (κ bits / gate) AND	XOR	calls to H per gate: garbler AND	XOR	evaluator AND	XOR	assump.
unoptimized textbook Yao	8	8	4	4	2.5	2.5	PRF
Yao + point-permute [BMR90]	4	4	4	4	1	1	PRF
$4 \to 3$ row reduction [NPS99]	3	3	4	4	1	1	PRF
$4 \to 2$ row reduction [PSSW09]	2	2	4	4	1	1	PRF
free-XOR [KS08]	3	0	4	0	1	0	CCR
fleXOR [KMR14]	2	$\{0,1,2\}$	4	$\{0,2,4\}$	1	$\{0,1,2\}$	CCR
half-gates [ZRE15]	2	0	4	0	2	0	CCR
[GLNP15]	2	1	4	3	2	1.5	PRF
ours	**1.5**	**0**	≤ 6	0	≤ 3	0	CCR

Credit: Table taken from proceedings version of [RR'21].

Malicious Alice

What can a malicious garbler do?

Simple Generic Defense
Cut-and-choose

Rough sketch:

- Alice commits to q garbled gates and the randomness in generating them.
- Bob opens all but one instances, including all the labels, and check that gates are garbled correctly; if not, abort.
- Use the unopened GC to compute.

Note: Use of commitment is crucial:

How do we get soundness error: $2^{\Omega(q)}$?

Malicious Bob

What can a malicious evaluator do?

Next class:
Quantum Cryptography