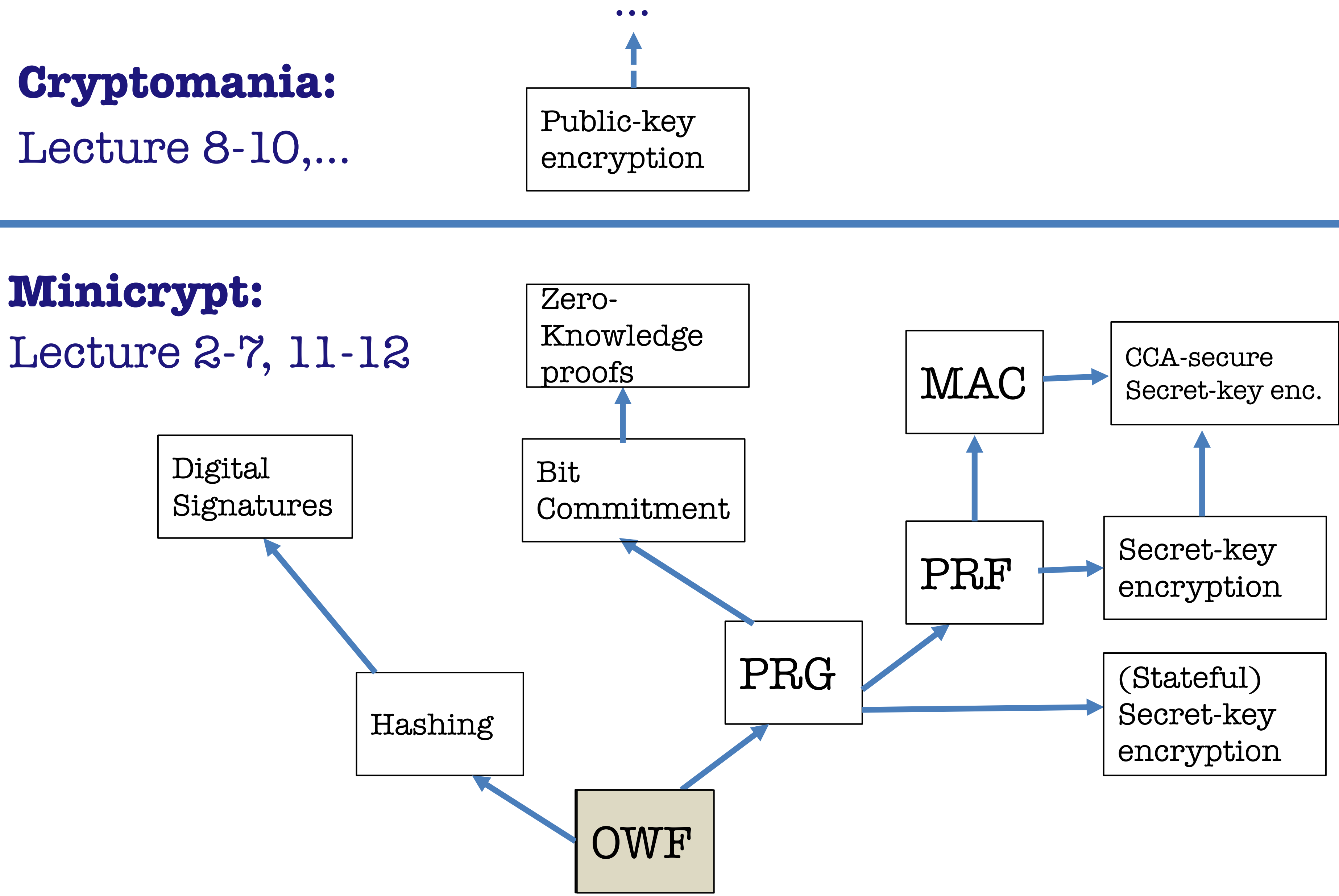
MIT 6.875

Foundations of Cryptography
Lecture 6

Roadmap of the Course:



This Week

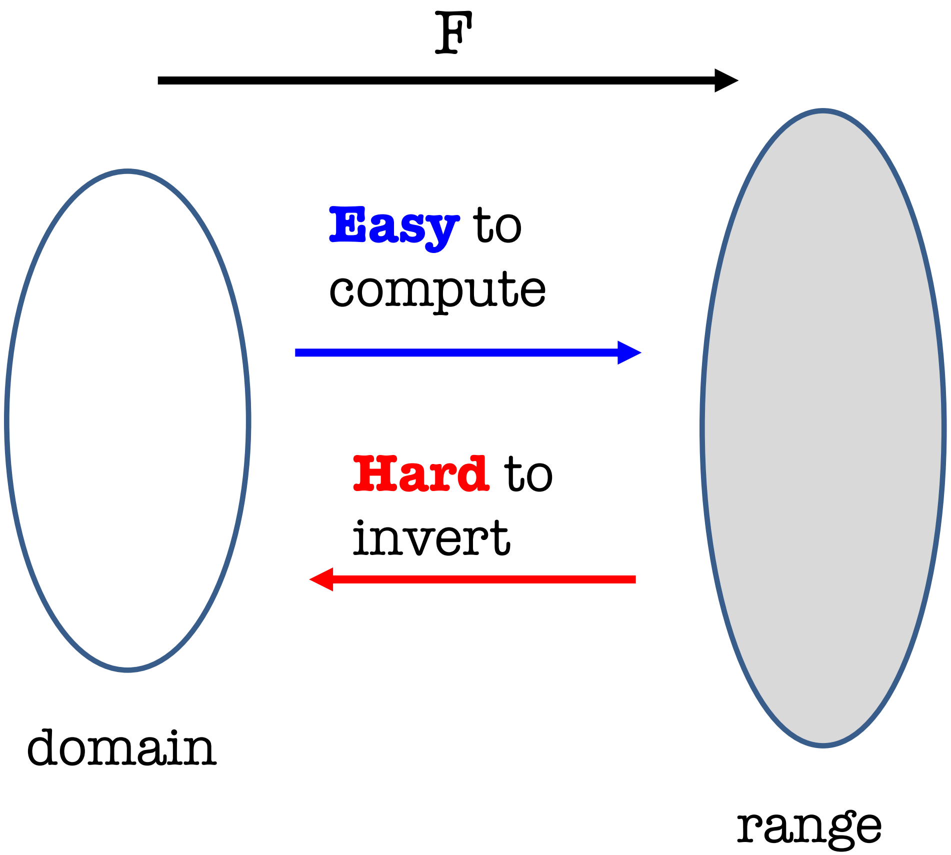
1. Define one-way functions (OWF).

2. Define Hardcore bits (HCB).

3. Show that one-way functions* + HCB $\Rightarrow$ PRG

4. **Goldreich-Levin Theorem**: every OWF has a HCB.

One-way Functions (Informally)



One-way Functions (Take 1)

A function (family) $\{F_n\}_{n\in\mathbb{N}}$ where $F_n\colon \{0,1\}^n \rightarrow \{0,1\}^{m(n)}$ is one-way if for every p.p.t. adversary A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; y = F_n(x) \colon A(1^n, y) = x] \leq \mu(n)$$

Consider $\boldsymbol{F_n(x) = 0}$ for all x.

This is one-way according to the above definition.
In fact, impossible to find *the* inverse even if A has unbounded time.

Conclusion: not a useful/meaningful definition.

One-way Functions (Take 1)

A function (family) $\{F_n\}_{n\in\mathbb{N}}$ where $F_n\colon \{0,1\}^n \rightarrow \{0,1\}^{m(n)}$ is one-way if for every p.p.t. adversary A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; y = F_n(x) \colon A(1^n, y) = x] \leq \mu(n)$$

The Right Definition: Impossible to find ***an*** inverse in p.p.t.

One-way Functions: The Definition

A function (family) $\{F_n\}_{n \in \mathbb{N}}$ where $F_n: \{0,1\}^n \rightarrow \{0,1\}^{m(n)}$ is one-way if for every p.p.t. adversary A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; y = F_n(x); A(1^n, y) = \boldsymbol{x'}: \boldsymbol{y} = \boldsymbol{F_n(x')}] \leq \mu(n)$$

- Can always find *an* inverse with unbounded time
- … but should be hard with probabilistic polynomial time

One-way Permutations:

One-to-one one-way functions with $m(n) = n$.

Today

1. Define one-way functions (OWF).

2. Define Hardcore bits (HCB).

3. Show that one-way *permutations* (OWP) $\Rightarrow$ PRG

4. Goldreich-Levin Theorem: every OWF has a HCB.

Hardcore Bits

If F is a one-way function, we know it's hard to compute a pre-image of $F(x)$ for a randomly chosen x.

How about computing partial information about an inverse?

Exercise: There are one-way functions for which it is easy to compute the first half of the bits of an inverse.

Hardcore Bits

If F is a one-way function, we know it's hard to compute a pre-image of $F(x)$ for a randomly chosen x.

HARDCORE BIT (Take 1)

Nevertheless, there has to be a hardcore set of hard to invert inputs. Concretely: Does there exist <u>some bit of x</u> that is <u>hard to compute</u>?

Does there necessarily exist <u>some bit of x</u> that is <u>hard to guess with probability non-negligibly better than 1/2</u>?

- Any bit can be guessed correctly w.p. 1/2
- So, "hard to compute" → "hard to guess with probability non-negligibly better than 1/2"

Hardcore Bits

If F is a one-way function, we know it's hard to compute a pre-image of $F(x)$ for a randomly chosen x.

HARDCORE BIT (Take 1)

For any function (family) $F: \{0,1\}^n \rightarrow \{0,1\}^m$, a bit $i = i(n)$ is hardcore if for every p.p.t. adversary A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; y = F(x): A(y) = x_i] \leq \frac{1}{2} + \mu(n)$$

Does every one-way function have a hardcore bit?

PS2: There are functions that are one-way, yet *every* bit is somewhat easy to predict (say, with probability $\frac{1}{2} + 1/n$).

So, we will generalize the notion of a hardcore "bit".

Hardcore Bits

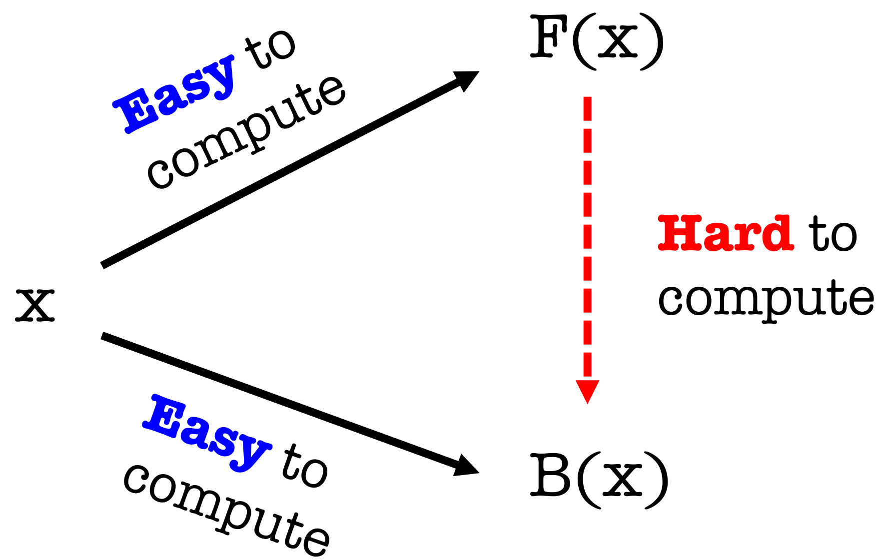
HARDCORE PREDICATE (Definition)

For any function (family) $F: \{0,1\}^n \to \{0,1\}^m$, a function $B: \{0,1\}^n \to \{0,1\}$ is a hardcore **predicate** if for every p.p.t. adversary A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; y = F(x): A(y) = B(x)] \leq \frac{1}{2} + \mu(n)$$

For us, henceforth, a hardcore bit will mean a hardcore predicate.

Hardcore Predicate (in pictures)



Discussion on the Definition

HARDCORE PREDICATE (Definition)

For any function (family) $F: \{0,1\}^n \rightarrow \{0,1\}^m$, a bit $B: \{0,1\}^n \rightarrow \{0,1\}$ is a hardcore **predicate** (HCP) if for every p.p.t. adversary A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; y = F(x): A(y) = B(x)] \leq \frac{1}{2} + \mu(n)$$

1. Definition of HCP makes sense for *any* function family, not just one-way functions.

2. Some functions can have information-theoretically hard to guess predicates (e.g., compressing functions)

3. We'll be interested in settings where x is uniquely determined given $\mathrm{F}(x)$, yet $\mathrm{B}(x)$ is hard to predict given $\mathrm{F}(x)$

Today

1. Define one-way functions (OWF).

2. Define Hardcore bits (HCB).

3. Show that one-way *permutations* (OWP) $\Rightarrow$ PRG

4. <u>Goldreich-Levin Theorem</u>: every OWF has a HCB.

OWP $\Rightarrow$ PRG

CONSTRUCTION

Let F be a one-way permutation, and B an associated hardcore predicate for F.

Then, define $G(x) = \mathrm{F}(x) \mid \mathrm{B}(x)$.

Theorem: G is a PRG assuming F is a one-way permutation.

(Note that G stretches by one bit. We already know how to turn this into a G' that stretches to any poly number of bits.)

OWP $\Rightarrow$ PRG

CONSTRUCTION

Let F be a one-way permutation, and B an associated hardcore predicate for F.

Then, define $G(x) = \mathrm{F}(x) \mid \mathrm{B}(x)$.

Theorem: G is a PRG assuming F is a one-way permutation.

Proof (next slide): Use next-bit unpredictability.

OWP ⇒ PRG

Theorem: G is a PRG assuming F is a one-way permutation.

Proof: Assume for contradiction that G is not a PRG.
Therefore, there is a next-bit predictor D, and index i, and a polynomial function p such that

$$\Pr[x \leftarrow \{0,1\}^n; y = G(x): D(y_{1\ldots i-1}) = y_i] \geq \frac{1}{2} + 1/p(n)$$

Observation: The index i has to be $n+1$. Do you see why?

Hint: $G(x) = \mathrm{F}(x) \mid \mathrm{B}(x)$ and F is a one-way permutation.

OWP $\Rightarrow$ PRG

Theorem: G is a PRG assuming F is a one-way permutation.

Proof: Assume for contradiction that G is not a PRG.
Therefore, there is a next-bit predictor D and a polynomial function p such that

$$\Pr[x \leftarrow \{0,1\}^n; y = G(x): D(y_{1\dots n}) = y_{n+1}] \geq \frac{1}{2} + 1/p(n)$$

OWP ⇒ PRG

Theorem: G is a PRG assuming F is a one-way permutation.

Proof: Assume for contradiction that G is not a PRG.
Therefore, there is a next-bit predictor D and a polynomial function p such that

$$\Pr[x \leftarrow \{0,1\}^n : D(F(x)) = B(x)] \geq \frac{1}{2} + 1/p(n)$$

So, D is a hardcore bit predictor! QED.

Today

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A Hardcore Predicate for all OWF

Let's shoot for a *universal* hardcore predicate.

i.e., a single predicate B where it is hard to guess $B(x)$ given $\mathrm{F}(x)$

Is this possible?

Turns out the answer is "no".

You will tell me why in PS2.

So, what is one to do?

Goldreich-Levin (GL) Theorem

Let $\{B_r: \{0,1\}^n \to \{0,1\}\}$ where

$$B_r(x) = \langle r, x \rangle = \sum_{i=1}^{n} r_i x_i \bmod 2$$

be a collection of predicates (one for each r). Then, a ***random*** B_r is hardcore for ***every*** one-way function F. That is, for every one-way function F, every PPT A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; r \leftarrow \{0,1\}^n: A(F(x), r) = B_r(x)] \leq \frac{1}{2} + \mu(n)$$

<u>Alternative Interpretation 1</u>: For every one-way function F, there is a related one-way function $F'(x, r) = (F(x), r)$ which has a *deterministic* hardcore predicate.

Goldreich-Levin (GL) Theorem

Let $\{B_r: \{0,1\}^n \to \{0,1\}\}$ where

$$B_r(x) = \langle r, x \rangle = \sum_{i=1}^{n} r_i x_i \bmod 2$$

be a collection of predicates (one for each r). Then, a ***random*** B_r is hardcore for ***every*** one-way function F. That is, for every one-way function F, every PPT A, there is a negligible function μ s.t.

$$\Pr[x \leftarrow \{0,1\}^n; r \leftarrow \{0,1\}^n: A(F(x), r) = B_r(x)] \leq \frac{1}{2} + \mu(n)$$

Alternative Interpretation 2: For every one-way function F, there ***exists*** (non-uniformly) a (possibly different) hardcore predicate $\langle r_F, x \rangle$. **(My favorite open problem: remove the non-uniformity)**

Proof of GL Theorem

Let's make our lives easier: assume a perfect predictor P

~~Assume for contradiction there is a predictor P~~

$$\Pr[x \leftarrow \{0,1\}^n; r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x\rangle] \geq \frac{1}{2} + 1/p(n)$$

$$\Pr[x \leftarrow \{0,1\}^n; r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x\rangle] = 1$$

We will need to show an inverter A for F

$$\Pr\left[x \leftarrow \{0,1\}^n : A\big(F(x)\big) = x' : F(x') = F(x)\right] \geq 1/p'(n)$$

Proof of GL Theorem

Let's make our lives easier: assume a perfect predictor P

~~Assume for contradiction there is a predictor P~~

$$\Pr[x \leftarrow \{0,1\}^n; r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x\rangle] = 1$$

The inverter A works as follows:

On input $\mathrm{y} = F(x)$, A runs the predictor P n times, on inputs $(y, e_1), (y, e_2), \ldots,$ and (y, e_n) where $e_1 = 100..0$, $e_2 = 010 \ldots 0$,… are the unit vectors.

Since A is perfect, it returns $\langle e_i, x\rangle = x_i$, the i^{th} bit of x on the i^{th} invocation.

Proof of GL Theorem

OK, now let's assume less: assume a pretty good predictor P

~~Assume for contradiction there is a predictor P~~

$$\Pr[x \leftarrow \{0,1\}^n; r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x\rangle] \geq \frac{3}{4} + 1/p(n)$$

First, we need an **averaging argument**.

Claim: For at least a $1/2p(n)$ fraction of the x,

$$\Pr[r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x\rangle] \geq \frac{3}{4} + 1/2p(n)$$

Proof: Exercise in counting.

Call these the good x.

Proof of GL Theorem

For at least a $1/2p(n)$ fraction of the x,

$$\Pr[r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x \rangle] \geq \frac{3}{4} + 1/2p(n)$$

Key Idea: Linearity

Pick a random r and ask P to tells us $\langle r, x \rangle$ and $\langle r + e_i, x \rangle$.
Subtract the two answers to get $\langle e_i, x \rangle = x_i$.

Proof: $\Pr[\text{we compute } x_i \text{ correctly}]$

$$\geq \Pr[\text{P predicts } \langle r, x \rangle \text{ and } \langle r + e_i, x \rangle \text{ correctly}]$$
$$= 1 - \Pr[\text{P predicts} \langle r, x \rangle \text{ or } \langle r + e_i, x \rangle \text{ wrong}]$$
$$\geq 1 - (\Pr[\text{P predicts} \langle r, x \rangle \text{ wrong}] + \Pr[\text{P predicts} \langle r + e_i, x \rangle \text{ wrong}])$$ *(by union bound)*
$$\geq 1 - 2 \cdot \left(\frac{1}{4} - \frac{1}{2p(n)}\right) = \frac{1}{2} + 1/p(n)$$

Proof of GL Theorem

For at least a $1/2p(n)$ fraction of the x,

$$\Pr[r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x \rangle] \geq \frac{3}{4} + 1/2p(n)$$

Inverter A:

Repeat for each $i \in \{1,2,\ldots,n\}$:

Repeat $\log n * p(n)$ times:

Pick a random r and ask P to tells us $\langle r, x \rangle$ and $\langle r + e_i, x \rangle$.
Subtract the two answers to get a guess for x_i.

Compute the majority of all such guesses and set the bit as x_i

Output the concatenation of all x_i as x.

Analysis: Chernoff + Union Bound

Real Proof (next lecture)

Assume (after averaging) that for $\geq 1/2p(n)$ fraction of the x,

$$\Pr[r \leftarrow \{0,1\}^n : P(F(x), r) = \langle r, x \rangle] \geq \mathbf{\frac{1}{2}} + 1/2p(n)$$

Key Idea: Pairwise independence

Reference: Goldreich Book Part 1, Section 2.5.2.

http://www.wisdom.weizmann.ac.il/~oded/PSBookFrag/part2N.ps

The Coding-Theoretic View of GL

$x \to (\langle x, r \rangle)_{r \in \{0,1\}^n}$ can be viewed as a highly redundant, exponentially long encoding of x = **the Hadamard code**.

$P(F(x), r)$ can be thought of as providing access to a **noisy** codeword.

What we proved = **unique decoding** algorithm for Hadamard code with error rate $\frac{1}{4} - 1/p(n)$.

The real proof = **list-decoding algorithm** for Hadamard code with error rate $\frac{1}{2} - 1/p(n)$.

Recap

1. Defined one-way functions (OWF).

2. Defined Hardcore bits (HCB).

3. Goldreich-Levin Theorem: every OWF has a HCB.

   (showed proof for an important special case)

4. Show that one-way *permutations* (OWP) $\Rightarrow$ PRG

   (*in fact, one-way functions* $\Rightarrow$ PRG, but that's a much harder theorem)