# MIT 6.875

## Foundations of Cryptography
## Lecture 15

# Zero Knowledge Proofs

# ZK Definition

An Interactive Protocol (P,V) is **perfect zero-knowledge** for a language $L$ if **for every PPT $V^*$**, there exists a (expected) poly time simulator S s.t. for every $x \in L$, the following two distributions are identical:

1. $view_{V^*}(P, V^*)$ 2. $S(x, 1^\lambda)$

Analogously:

statistical and computational zero-knowledge

# Zero Knowledge Interactive Proof for QR

$\mathcal{L} = \{(N, y): y \text{ is a quadratic residue mod } N\}$.

$s = r^2 \pmod{N}$ →

$(N, y)$

← $b \leftarrow \{0,1\}$

$(N, y)$

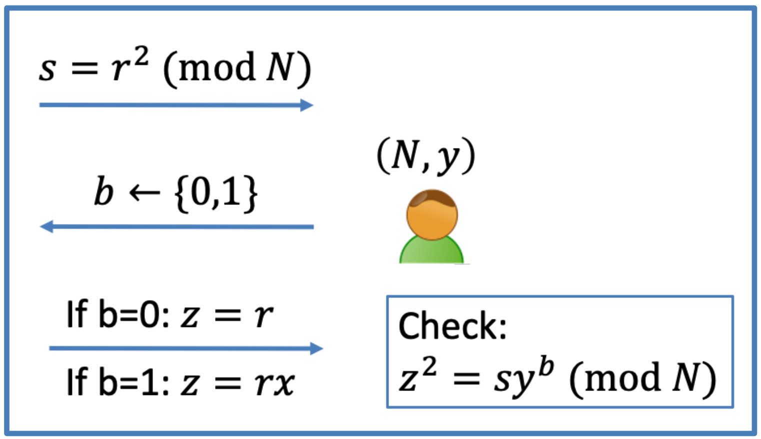
If b=0: $z = r$ →

If b=1: $z = rx$

Check:
$z^2 = sy^b \pmod{N}$

# We Proved:

**Thm:** The QR protocol is **honest verifier** zero knowledge.




$view_V(P, V)$:
$(s, b, z)$

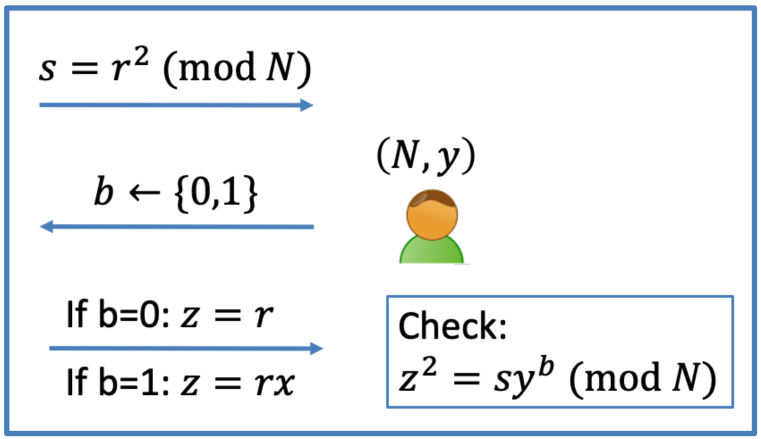
**Simulator S works as follows:**

1. First pick a random bit b.
2. pick a random $z \in Z_N^*$.
3. compute $\mathrm{s} = z^2/y^b$.
4. output $(\mathrm{s}, \mathrm{b}, \mathrm{z})$.

**Claim:** The simulated transcript is identically distributed as the real transcript in the interaction (P,V).

# NOW: (Malicious Ver) Zero Knowledge

**Theorem:** The QR protocol is (malicious verifier) zero knowledge.




$$view_{V^*}(P, V^*):$$
$$(s, b, z)$$

**Simulator S works as follows:**

1. First pick a random s and "feed it to" $V^*$.

2. Let $\mathrm{b} = V^*(s)$.

Now what???

# (Malicious Ver) Zero Knowledge

**Theorem:** The QR protocol is (malicious verifier) zero knowledge.

**Simulator S works as follows:**

1. First set $s = \frac{z^2}{y^b}$ for a random z and b and feed s to $V^*$.

2. Let $\mathrm{b}' = V^*(s)$.

3. If $b' = b$, output $(\mathrm{s}, \mathrm{b}, \mathrm{z})$ and stop.

4. Otherwise, go back to step 1 and repeat. (also called “rewinding”).

**Simulator S works as follows:**

1. First set $s = \frac{z^2}{y^b}$ for a random z and feed s to $V^*$.

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3. If $b' = b$, output $(\mathrm{s}, \mathrm{b}, \mathrm{z})$ and stop.

4. Otherwise, go back to step 1 and repeat. (also called "rewinding").

**Lemma**:

(1) S runs in expected polynomial-time.

(2) When S outputs a view, it is identically distributed to the view of $V^*$ in a real execution.

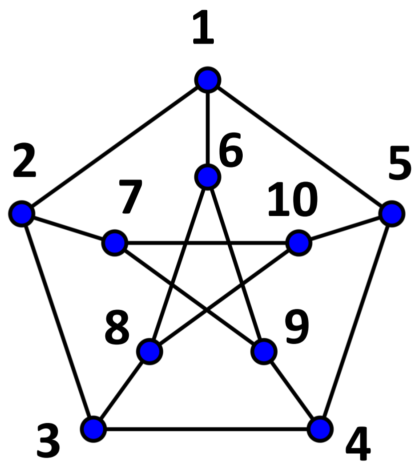
# What Made it Possible?

1. **Each statement had multiple proofs** of which the prover chooses one at random.

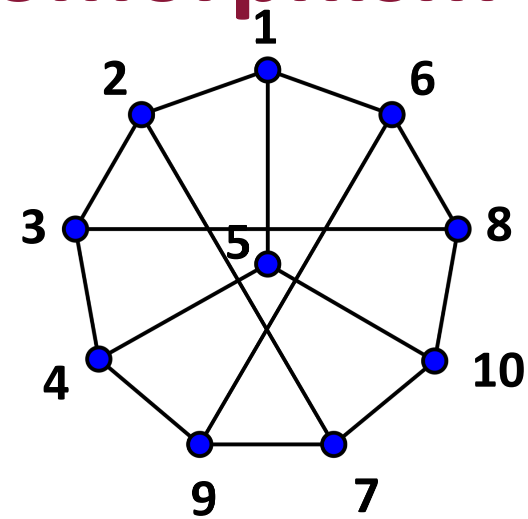
2. **Each such proof is made of two parts**: seeing either one on its own gives the verifier no knowledge; seeing both imply 100% correctness.

3. **Verifier chooses to see either part, at random**. The prover's ability to provide either part on demand convinces the verifier.

# ZK Proof for Graph Isomorphism




Graph G




Graph H

$\mathbf{H} = \boldsymbol{\pi}(\boldsymbol{G})$

**Prover**

$K = \rho(G)$

where $\rho$ is a random permutation

**Verifier**

random challenge bit $b$

$b = 0$: send $\pi_0$ s.t. $\mathrm{K} = \pi_0(G)$

$b = 1$: send $\pi_1$ s.t. $\mathrm{H} = \pi_1(K)$

# ZK Proof for Graph Isomorphism

**Completeness**: Exercise.

$\mathbf{H} = \boldsymbol{\pi}(\boldsymbol{G})$


$K = \rho(G)$

where $\rho$ is a random permutation

random challenge bit $b$


$b = 0$: send $\pi_0 = \rho$

$b = 1$: send $\pi_1 = \pi \circ \rho^{-1}$

# ZK Proof for Graph Isomorphism

**Soundness**: Suppose G and H are non-isomorphic, and a prover could answer both the verifier challenges. Then, $\mathrm{K} = \pi_0(G)$ and $\mathrm{H} = \pi_1(K)$.

In other words, $\mathrm{H} = \pi_1 \circ \pi_0(G)$, a contradiction!

$\mathbf{H} = \boldsymbol{\pi}(\boldsymbol{G})$

Prover

$K = \rho(G)$ →

where $\rho$ is a random permutation

random challenge bit $b$ ←

Verifier

$b = 0$: send $\pi_0 = \rho$ →

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# ZK Proof for Graph Isomorphism

**Zero Knowledge**: Exercise.

$\mathbf{H} = \boldsymbol{\pi}(\boldsymbol{G})$

Prover

$K = \rho(G)$

where $\rho$ is a random permutation

random challenge bit $b$

$b = 0$: send $\pi_0 = \rho$

$b = 1$: send $\pi_1 = \pi \circ \rho^{-1}$

Verifier

# Efficient Prover (given a Witness)

In both these protocols, the (honest) prover is actually polynomial-time ***given the NP witness*** (the square root of $y$ in the case of QR, and the isomorphism in the case of graph-iso.)

Soundness is nevertheless against any, even computationally unbounded, prover $P^*$.

# Do all NP languages have Perfect ZK proofs?

We showed two languages with perfect ZK proofs. Can we show this for *all* NP languages?

**Theorem** [Fortnow'89, Aiello-Hastad'87] No, unless bizarre stuff happens in complexity theory (technically: the polynomial hierarchy collapses.)

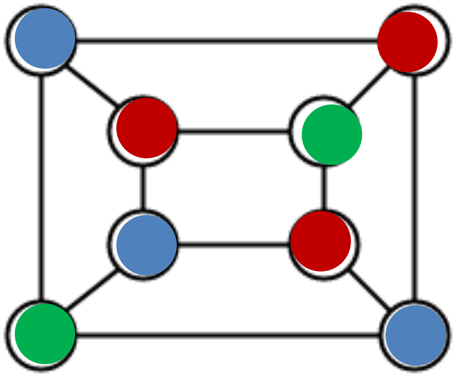
# Do all NP languages have ZK proofs?

**Nevertheless, today, we will show:**

<u>**Theorem**</u> [Goldreich-Micali-Wigderson'87] Assuming one-way functions exist, all of NP has computational zero-knowledge proofs.

*This theorem is amazing*: it tells us that everything that can be proved (in the sense of Euclid) can be proved in zero knowledge!

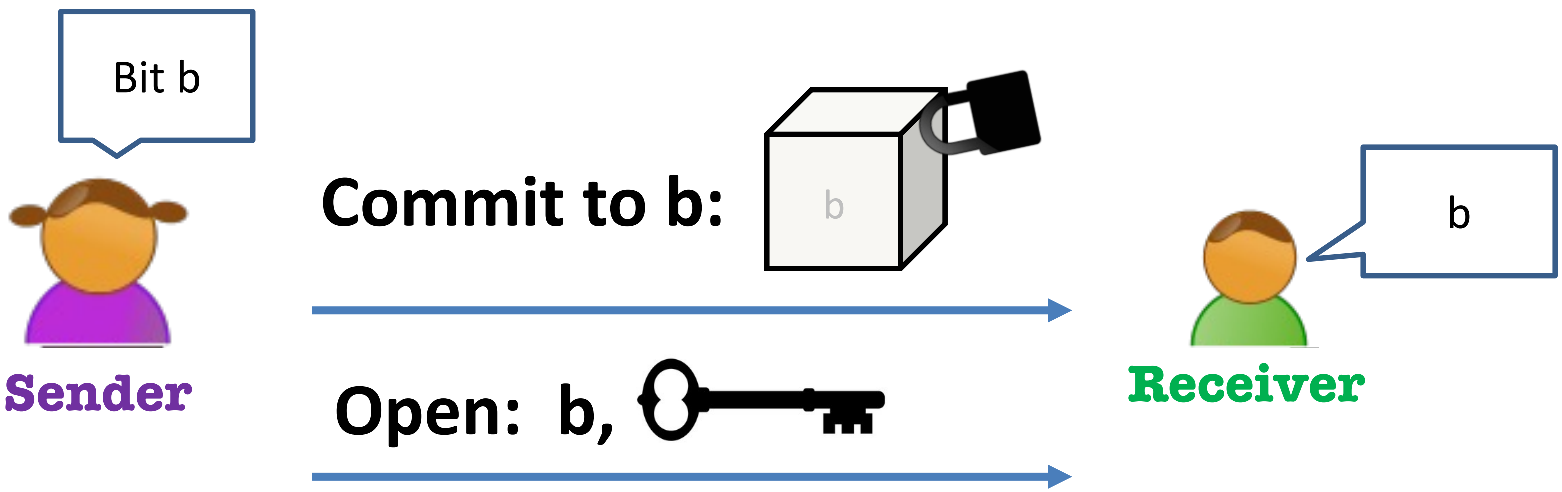
# Zero Knowledge Proof for 3-Coloring



## ***NP-Complete*** **Problem:**

Every other problem in NP can be reduced to it.

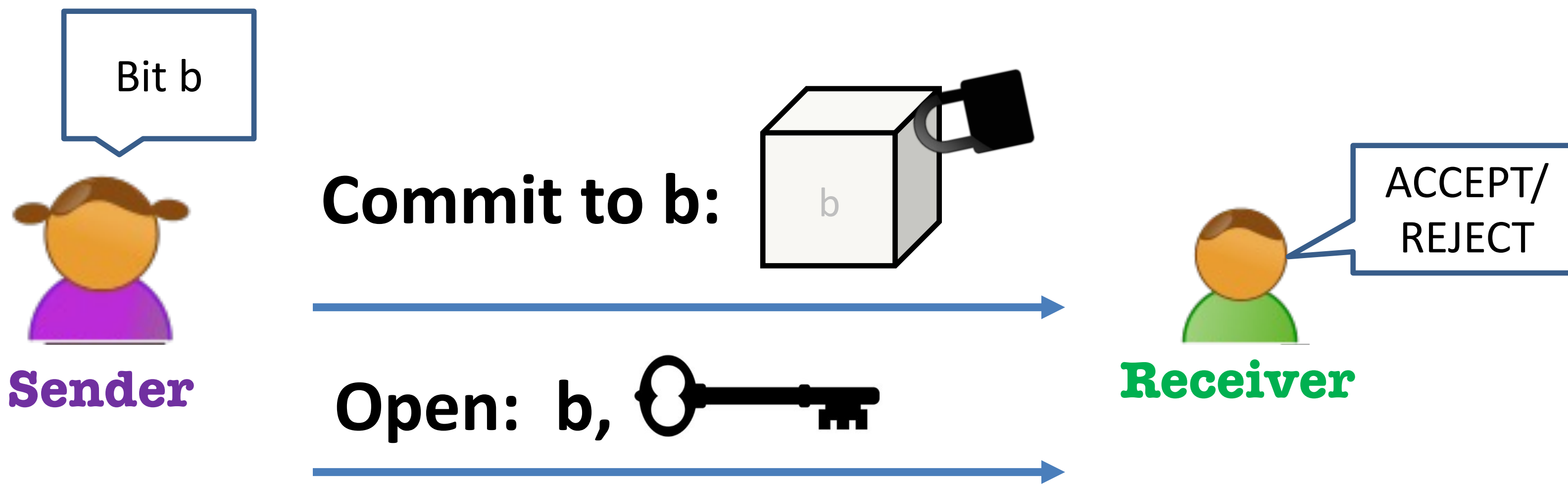
We need a commitment scheme (aka a "locking scheme" from pset 1).




1. **Hiding:** The locked box should completely hide b.

2. **Binding:** Sender shouldn't be able to open to 1-b.

**In pset 1, you implemented a commitment scheme using PRGs. We will later show another construction using one-way permutations.**

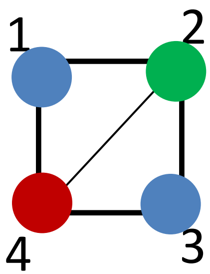


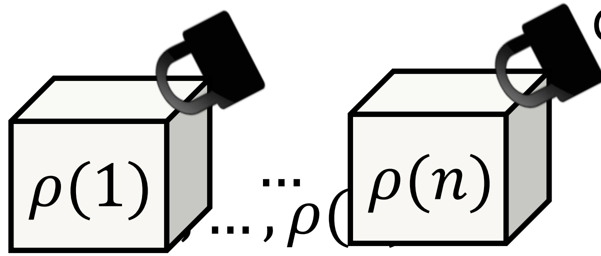

1. **Hiding:** The locked box should completely hide b.

2. **Binding:** Sender shouldn't be able to open to 1-b.

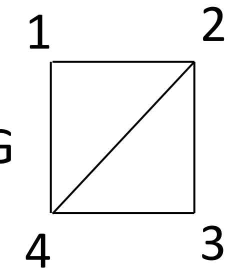
# Zero Knowledge Proof for 3COL

Graph G
=(V,E)




Graph G




random edge $(i, j)$

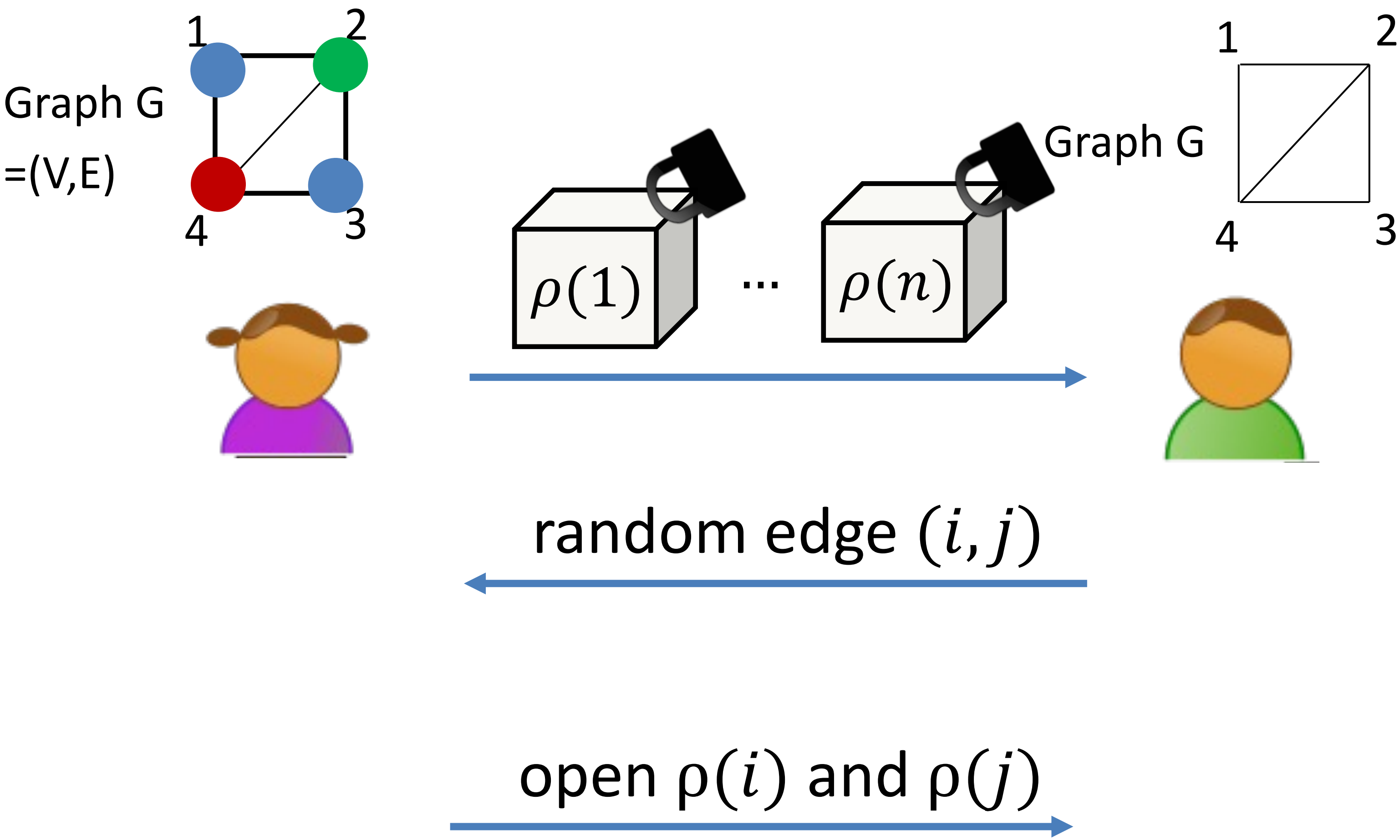
Come up with a random permutation of the colors

$$\rho: V \to \{R, B, G\}$$

open $\rho(i)$ and $\rho(j)$

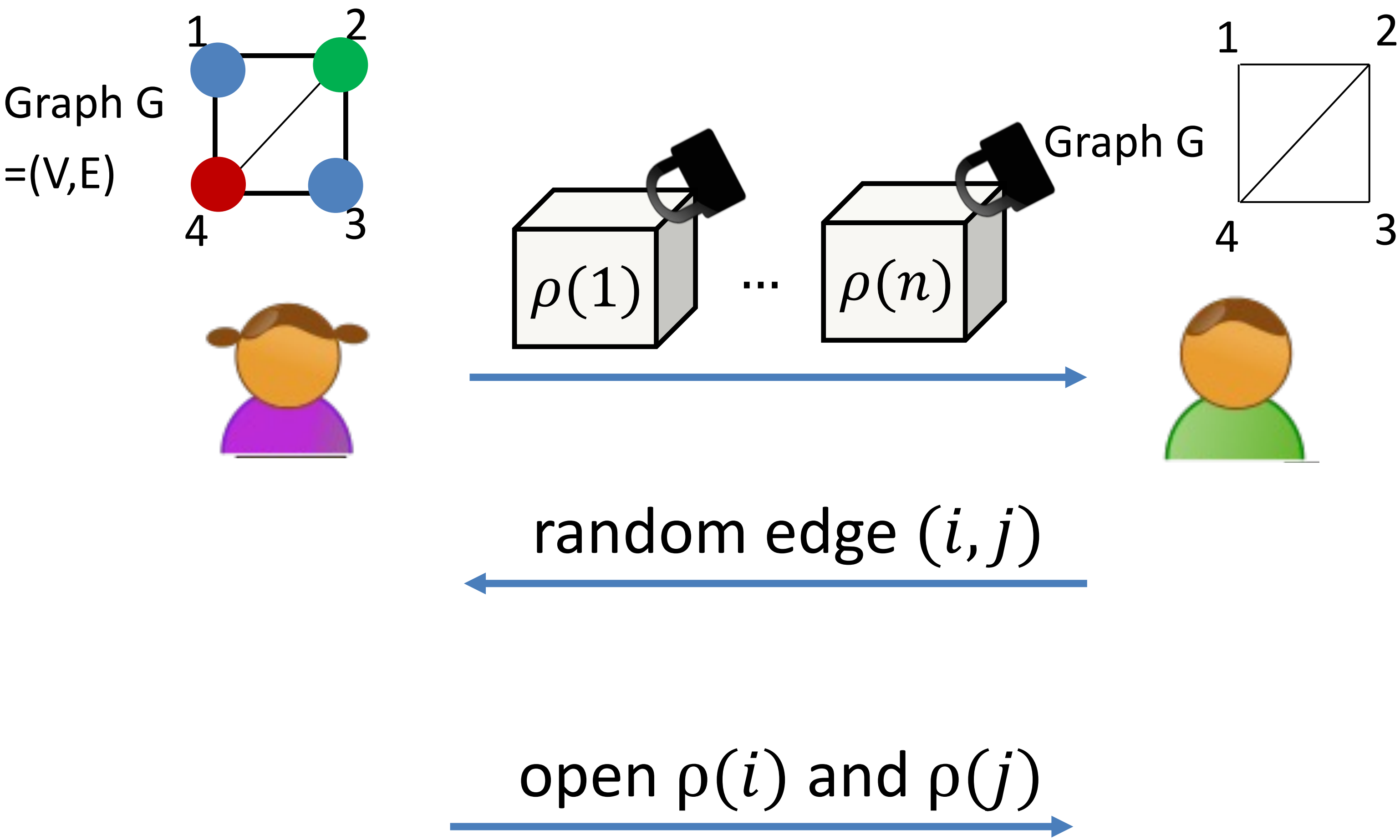
1. Check the openings
2. Check: $\rho(i), \rho(j) \in \{R, B, G\}$
3. Check: $\rho(i) \neq \rho(j)$ .

# Zero Knowledge Proof for 3COL



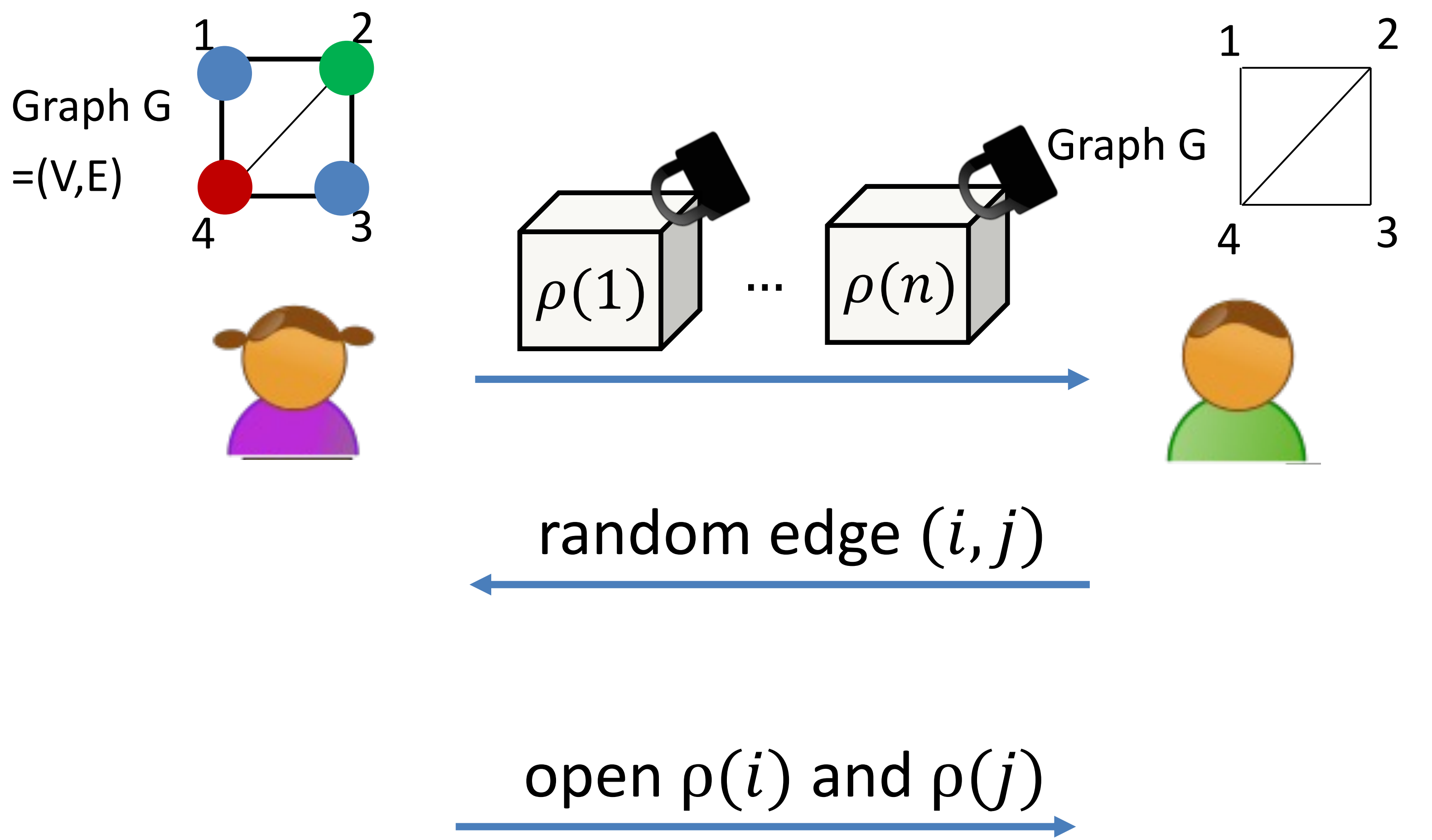

**Completeness**: Exercise.

# Zero Knowledge Proof for 3COL




**Soundness**: If the graph is not 3COL, in every 3-coloring (that P commits to), there is some edge whose end-points have the same color. V will catch this edge and reject with probability $\geq 1/|E|$.

# Zero Knowledge Proof for 3COL




**Repeat $|\boldsymbol{E}| \cdot \boldsymbol{\lambda}$** times to get the verifier to accept with probability

$$\leq (1 - 1/|E|)^{|E| \cdot \lambda} \leq 2^{-\lambda}$$

# Commitment Schemes

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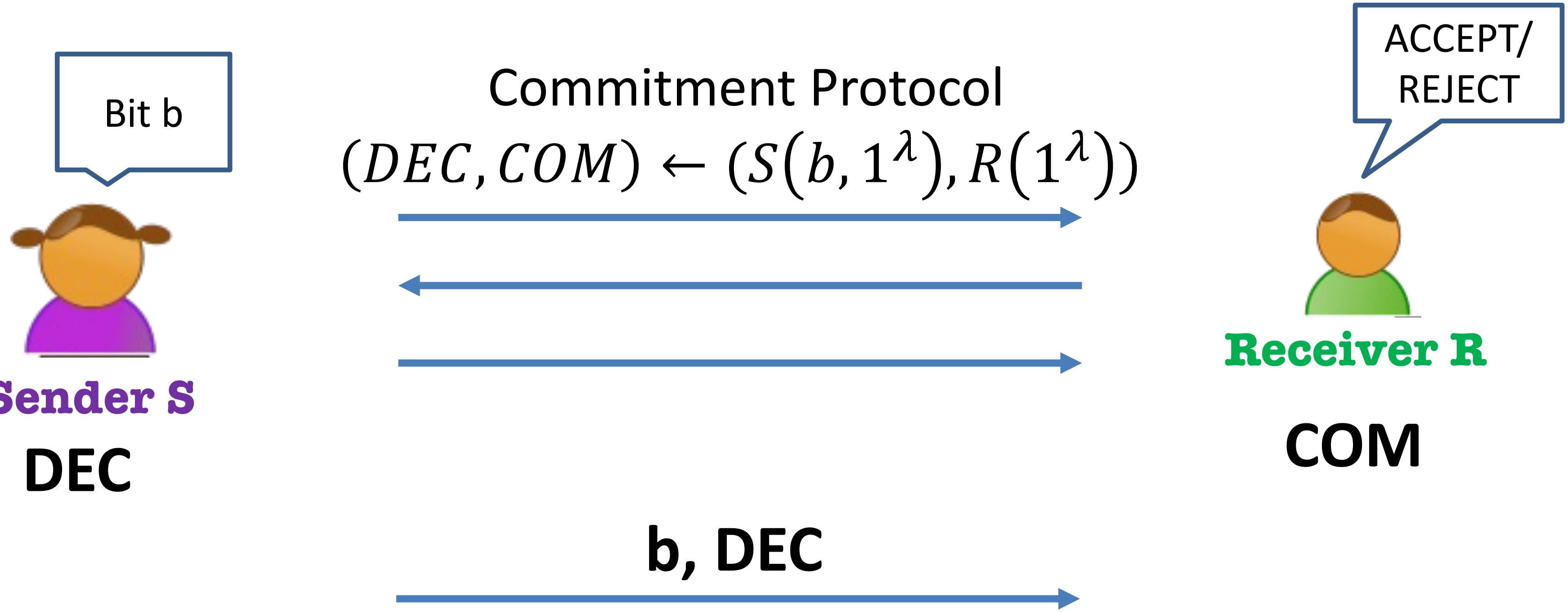
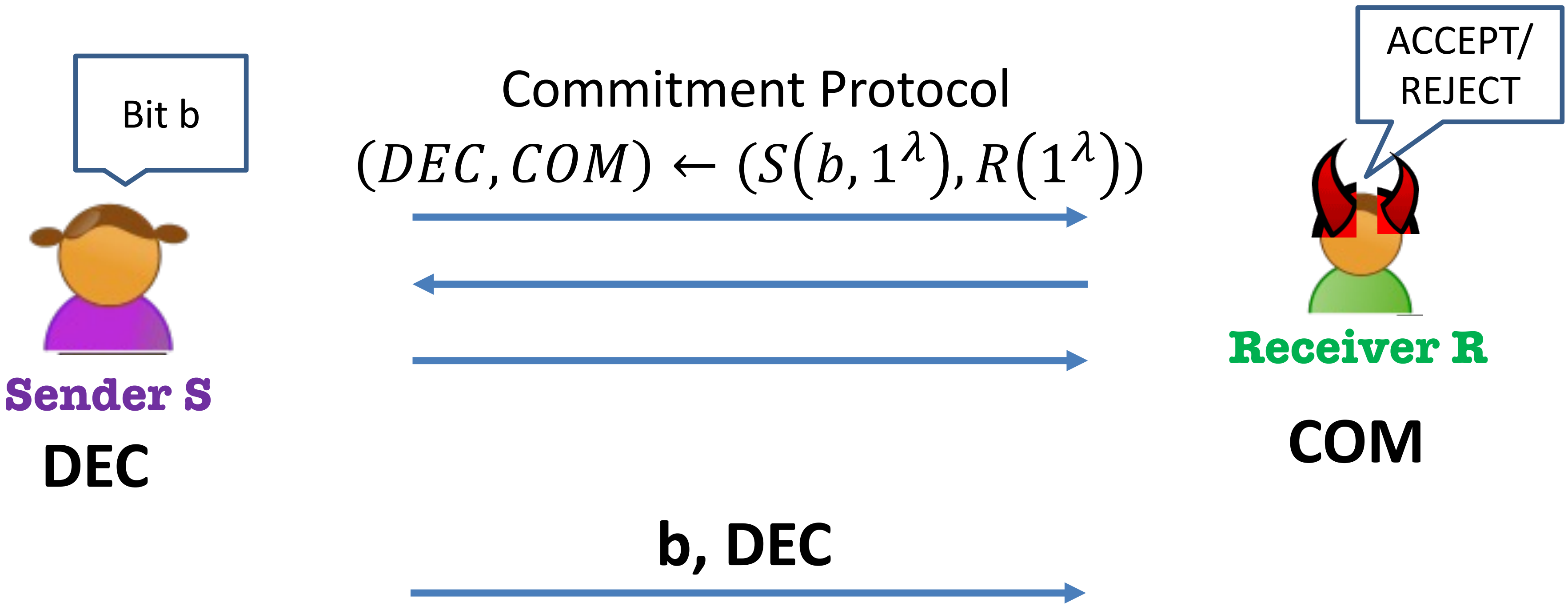



**1. Completeness:** R always accepts in an honest execution.

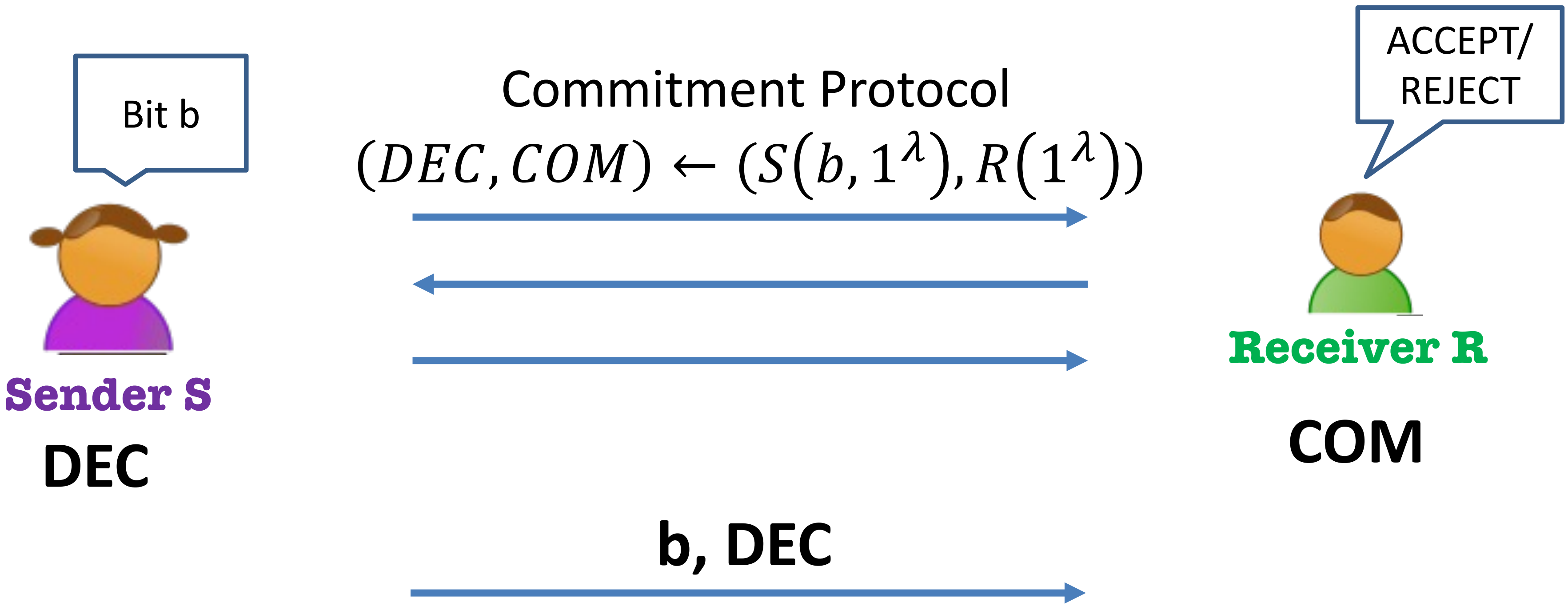
# Commitment Schemes




**2. Computational Hiding:** For every possibly malicious (PPT) $R^*$,

$$view_{R^*}(S(0), R^*) \approx_c view_{R^*}(S(1), R^*)$$

# Commitment Schemes




**3. Perfect Binding:** For every possibly malicious $S^*$, let COM be the receiver's output in an execution of $(S^*, R)$. There is no pair of decommitments $(DEC_0, DEC_1)$ s.t. R accepts both $(\text{com}, 0, DEC_0)$ and $(\text{com}, 1, DEC_1)$.

# A Commitment Scheme from any OWP

Bit b

**Sender S**

$DEC = r$

$COM = (f(r), HCB(r) \oplus b)$

$OPEN: (b, r)$

**Receiver R**

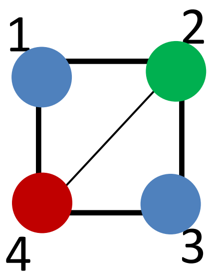
Let $COM = (x, y)$.
Check that
1. $f(r) = x$ and
2. $HCB(r) \oplus b = y$

1. **Completeness:** Exercise.
2. **Comp. Hiding:** by the hardcore bit property.
3. **Perfect Binding:** because f is a permutation.

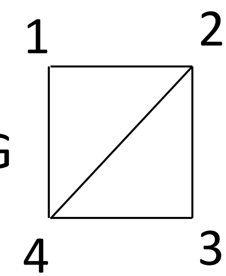
# Back to ZK Proof for 3COL

Graph G

=(V,E)




Graph G




$\{Com(\rho(k); r_k)\}_{k=1}^{n}$

random edge $(i, j)$

send openings $\rho(i), r_i$ and $\rho(j), r_j$

# *Why is this zero-knowledge?*

**Simulator S works as follows:**

1. First pick a random edge $(i^*, j^*)$

   Color edge $(i^*, j^*)$ with random, different colors

   Color all other vertices red.

$\{Com(\rho(k); r_k)\}_{k=1}^{n}$

edge $(i, j)$

2. Feed the commitments of the colors to $V^*$ and get edge $(i, j)$

3. If $(i, j) \neq (i^*, j^*)$, go back and repeat.

send openings $r_i$ and $r_j$

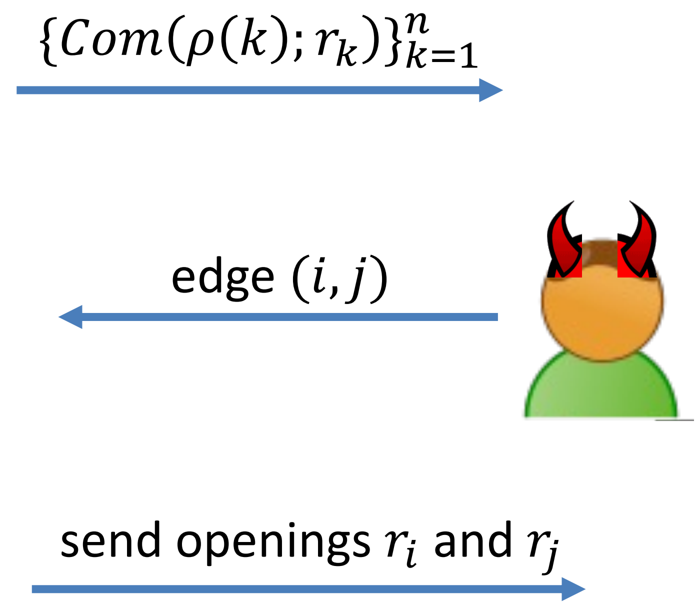
4. If $(i, j) = (i^*, j^*)$, output the commitments and openings $r_i$ and $r_j$ as the simulated transcript.

# *Why is this zero-knowledge?*

**Lemma**:

(1) Assuming the commitment is hiding, S runs in expected polynomial-time.

(2) When S outputs a view, it is comp. indist. from the view of $V^*$ in a real execution.



# *Why is this zero-knowledge?*

**Simulator S works as follows (call this Hybrid 0)**

1. First pick a random edge $(i^*, j^*)$

   Color edge $(i^*, j^*)$ with random, different colors

   Color all other vertices red.

$\xrightarrow{\{Com(\rho(k); r_k)\}_{k=1}^{n}}$

2. Feed the commitments of the colors to $V^*$ and get edge $(i, j)$

$\xleftarrow{\text{edge } (i,j)}$

3. If $(i, j) \neq (i^*, j^*)$, go back and repeat.

$\xrightarrow{\text{send openings } r_i \text{ and } r_j}$

4. If $(i, j) = (i^*, j^*)$, output the commitments and openings $r_i$ and $r_j$ as the simulated transcript.

# *Why is this zero-knowledge?*

## Not-a-Simulator S works as follows (call this Hybrid 1)

1. First pick a random edge $(i^*, j^*)$

   Permute a legal coloring and color all edges correctly.

$\{Com(\rho(k); r_k)\}_{k=1}^{n}$

edge $(i, j)$

2. Feed the commitments of the colors to $V^*$ and get edge $(i, j)$

3. If $(i, j) \neq (i^*, j^*)$, go back and repeat.

send openings $r_i$ and $r_j$

4. If $(i, j) = (i^*, j^*)$, output the commitments and openings $r_i$ and $r_j$ as the simulated transcript.

# *Why is this zero-knowledge?*

**Claim:** Hybrids 0 and 1 are computationally indistinguishable, assuming the commitment scheme is computationally hiding.

**Proof:** By contradiction. Show a reduction that breaks the hiding property of the commitment scheme, assuming there is a distinguisher between hybrids 0 and 1.

# *Why is this zero-knowledge?*

## Not-a-Simulator S works as follows (call this Hybrid 1)

1. First pick a random edge $(i^*, j^*)$

   Permute a legal coloring and color all edges correctly.

$\{Com(\rho(k); r_k)\}_{k=1}^{n}$

edge $(i, j)$

2. Feed the commitments of the colors to $V^*$ and get edge $(i, j)$

send openings $r_i$ and $r_j$

3. If $(i, j) \neq (i^*, j^*)$, go back and repeat.

4. If $(i, j) = (i^*, j^*)$, output the commitments and openings $r_i$ and $r_j$ as the simulated transcript.

# *Why is this zero-knowledge?*

## Here is the real view of V* (Hybrid 2)

1. ~~First pick a random edge $(i^*, j^*)$~~

   Permute a legal coloring and color all edges correctly.

$\{Com(\rho(k); r_k)\}_{k=1}^{n}$

edge $(i, j)$

2. Feed the commitments of the colors to $V^*$ and get edge $(i, j)$

send openings $r_i$ and $r_j$

3. ~~If $(i, j) \neq (i^*, j^*)$, go back and repeat.~~

4. ~~If $(i, j) = (i^*, j^*)$,~~ output the commitments and openings $r_i$ and $r_j$ as the transcript.

# *Why is this zero-knowledge?*

**Claim:** Hybrids 1 and 2 are identical.

Hybrid 1 merely samples from the same distribution as Hybrid 2 and, with probability $1 - 1/|E|$, decides to throw it away and resample.

# *Put together:*

**Theorem:** The 3COL protocol is zero knowledge.

# Examples of NP Assertions

- **My public key is well-formed** (e.g. in RSA, the public key is $N$, a product of two primes together with an e that is relatively prime to $\varphi(N)$.)
- **Encrypted bitcoin (or Zcash): "I have enough money to pay you."** (e.g. I will publish an encryption of my bank account and prove to you that my balance is $\geq \$X.$ )
- **Running programs on encrypted inputs:** Given Enc(x) and y, prove that y = PROG(x).

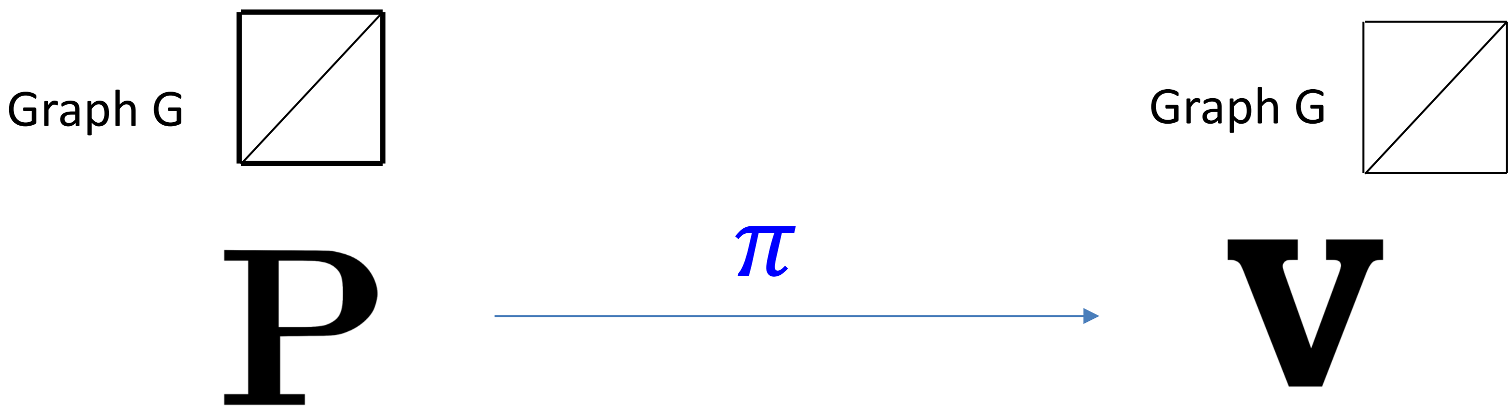
# Examples of NP Assertions

- **Running programs on encrypted inputs:** Given Enc(x) and y, prove that y = PROG(x).

**More generally: A tool to enforce honest behavior without revealing information.**

# Interaction is Necessary for ZK

Suppose there *were* a non-interactive ZK proof system for 3COL.

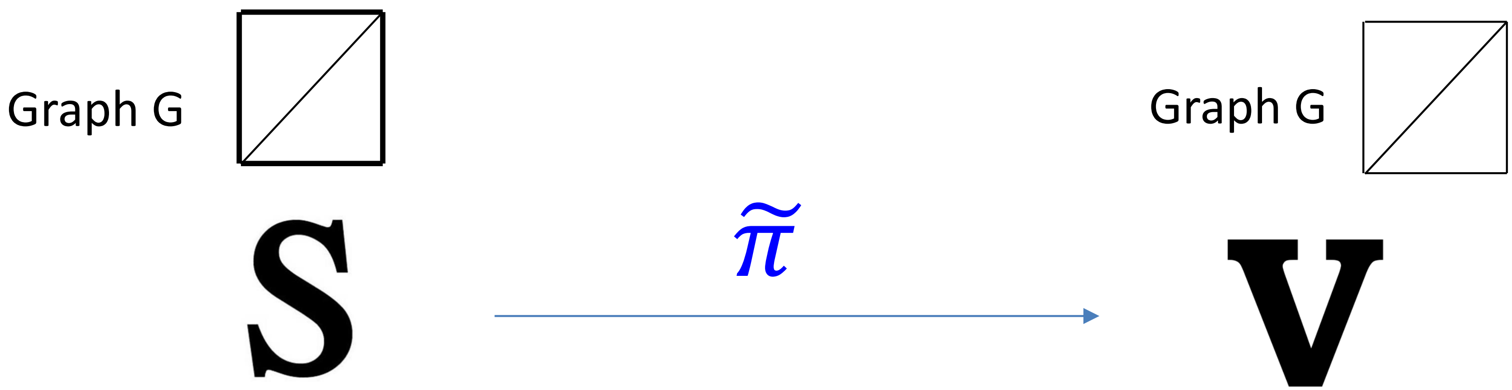



Step 1. When G is in 3COL, V accepts the proof $\pi$.

(Completeness)

# Interaction is Necessary for ZK

Suppose there *were* a non-interactive ZK proof system for 3COL.

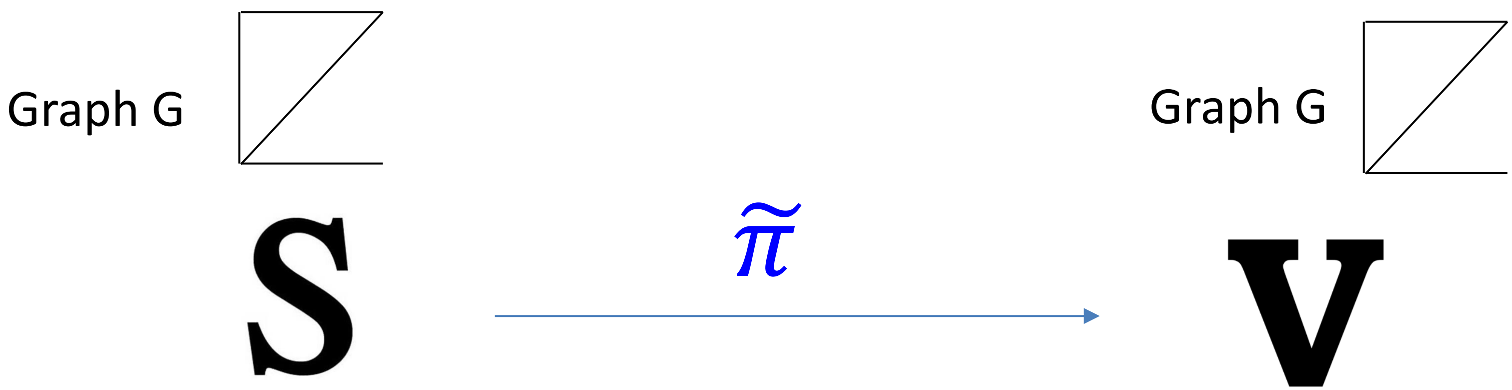



Step 2. **PPT** Simulator S, **given only G in 3COL**, produces an indistinguishable proof $\tilde{\pi}$ (Zero Knowledge).

**In particular, V accepts $\tilde{\pi}$.**

# Interaction is Necessary for ZK

Suppose there *were* a non-interactive ZK proof system for 3COL.



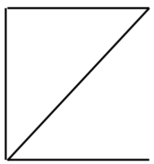

Step 3. Imagine running the Simulator S on a $G \notin$ 3COL. It produces a proof $\tilde{\pi}$ which the verifier still accepts!

**(WHY?! Because S and V are PPT. They together cannot tell if the input graph is 3COL or not)**

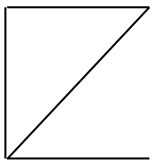
# Interaction is Necessary for ZK

Suppose there *were* a non-interactive ZK proof system for 3COL.

Graph G



Graph G



$$\mathbf{P} \xrightarrow{\pi} \mathbf{V}$$

Step 4. **Therefore, S is a cheating prover!**

Produces a proof for a $G \notin$ 3COL that the verifier nevertheless accepts.

**Ergo, the proof system is NOT SOUND!**

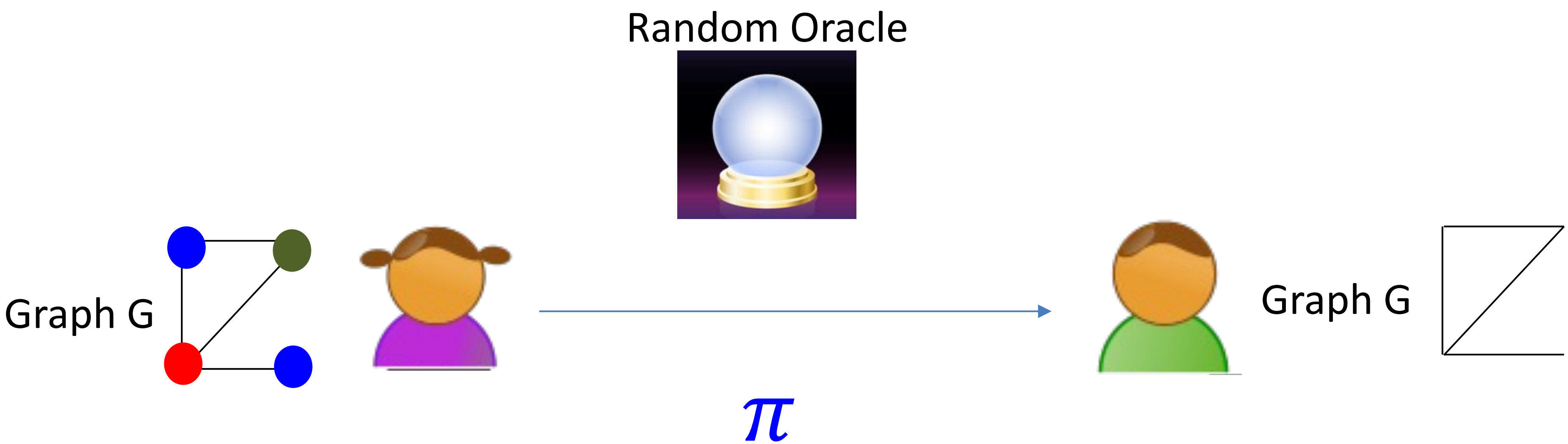
$\blacksquare$

**THE END**

***Or, is it?***

# Two Roads to Non-Interactive ZK (NIZK)

1. Random Oracle Model & Fiat-Shamir Transform.




2. Common Random String Model.